Handbook of Engineering Mathematics

BY

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113 ILLUSTRATIONS

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INTRODUCTION

BY

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In publishing this book the authors have endeavored to supply a handy means of reference to theoretical and applied mathematics used in engineering, and while the first aim has been to make this a mathematical handbook, the book is of greater value because it includes the underlying engineering data and applications as well as the mathematical formulæ.

It is intended primarily for students in engineering schools and colleges, and should serve as a convenient reminder of things which are easily forgotten but are likely to be needed in their later work.

In including differential equations, the authors have gone as far as seems necessary for students and engineers who have taken the ordinary undergraduate college course.

The increasing need of mathematics in engineering should assure this book a broad field of usefulness, not only to students in technical schools and colleges but also to practising engineers.

E. J. B.

TABLE OF CONTENTS

Engineering Mathematics

ALGEBRA

Quadratic Equations

$$ax^2 + bx + c = 0 \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The term $b^2 - 4ac$, called the **discriminant,** determines the nature of the roots. If b^2 is greater than $4ac$, the roots are real. If b^2 is less than $4ac$, the roots are imaginary. And if $b^2 = 4ac$, the roots are real and equal.

Exponents

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n}$$

$$a^m = \frac{1}{a^{-m}} \qquad a^{-m} = \frac{1}{a^m}$$

$$(a^m)^n = a^{mn}$$

Special and Indeterminate Forms

$$a^0 = 1$$

$$a^\infty = \infty, \qquad a > 1$$

$$a^{-\infty} = \frac{1}{a^\infty} = \frac{1}{\infty} = 0, \qquad a > 1$$

$$\frac{a}{0} = \infty \qquad \frac{a}{\infty} = 0$$

$$\frac{\infty}{a} = \infty \qquad \frac{0}{a} = 0$$

$0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, 1^∞, ∞^0, $\infty - \infty$ are indeterminate.

For the evaluation of indeterminate forms, see page 38.

Binomial Theorem

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Arithmetical Progression

An **arithmetical progression** is one whose terms increase or decrease by a common difference,

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots$$

the **last term** is $L = a + (n-1)d$

the **sum of the terms** is

$$S = \frac{n}{2}(a+L) = \frac{n}{2}[2a + (n-1)d]$$

a = first term

n = number of terms

d = common difference

Geometrical Progression

Quantities are in **geometrical progression** when each term is equal to the preceding term multiplied by a constant,

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots$$

the **last term** is $L = ar^{n-1}$

the **sum of the terms** is

$$S = \frac{a(r^n-1)}{r-1} = \frac{a(1-r^n)}{1-r} = \frac{rL-a}{r-1}$$

a = first term

r = constant ratio

n = number of terms

The **sum** of an **infinite number** of terms in geometrical progression is

$$S = \frac{a}{1 - r}$$

in which the ratio r must be less than 1 if the series is to be convergent (see Infinite Series).

Logarithms

The **logarithm** of any number to a given base is the power to which the base must be raised in order to produce the given number, thus:

if $\quad x^m = y, \quad$ then $\quad m = \log_x y,$

that is, m is the logarithm of y to the base x.

The following relations hold for any base:

$$\log ab = \log a + \log b$$

$$\log \frac{a}{b} = \log a - \log b$$

$$\log a^n = n \log a$$

$$\log \frac{1}{a} = -\log a$$

The **base of the common system** of logarithms is 10.

The **base of the natural system** of logarithms (also called Naperian or hyperbolic logarithms) is $e = 2.7182818284 \ldots$.

A logarithm may be transformed from any given base to any other desired base by the relation:

$$\log_b N = \frac{\log_a N}{\log_a b}.$$

To transform a logarithm **from base 10 to base e,**

multiply by 2.302585 . . . (where 2.302585 . . . is the logarithm of 10 to the base e):

$$\log_e a = 2.302585 \log_{10} a$$

To transform a logarithm **from base e to base 10,** divide by 2.302585:

$$\log_{10} a = \frac{1}{2.302585} \log_e a = 0.434294 \log_e a$$

Special forms:

$$\log 1 = 0 \qquad \text{(to any base)}$$
$$\log_a a = 1 \qquad \log_e e = 1$$
$$\log 0 = -\infty \qquad \log \infty = \infty$$

Cubic and Higher Degree Equations

The approximate values of the real roots of an algebraic equation containing only one variable may be found graphically.

For instance, let it be required to solve the equation $x^3 + Ax - B = 0$. This may be written as $x^3 = -Ax + B$, or as two simultaneous equations $y = x^3$ and $y = -Ax + B$. The graph of each of these equations being plotted, the abscissas of their points of intersection give the real roots of the cubic. The curve $y = x^3$ should be plotted on cross-section paper by the aid of a table of cubes. The curve $y = -Ax + B$ is the equation of a straight line, and is therefore determined by plotting two points.

Algebraic equations of any degree may be solved by Newton's method of approximation; see page 39.

Transcendental Equations

The graphic method given under Cubic and Higher Degree Equations is also applicable to many trans-

cendental equations. Thus, the equation $Ax - \sin x = 0$ may be solved by plotting the two simultaneous equations $y = Ax$ and $y = \sin x$. The curve $y = \sin x$ is readily plotted with the aid of a table of sines, while the other curve $y = Ax$ is a straight line passing through the origin.

Infinite Series

An **infinite series** is one containing an unlimited number of terms. Such a series is **convergent** if the sum of its terms is a finite quantity. It is **divergent** when the sum of its terms does not approach a finite limit.

Comparison Test. A series is converging if each term in it is equal to or less than the corresponding term of a known converging series.

Converging series for comparison:

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots \qquad [r < 1]$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} + \cdots$$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots$$

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots \qquad [p > 1]$$

A series is diverging if each term in it is equal to or greater than the corresponding term of a known diverging series.

Diverging series for comparison:

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots \qquad [r \geqq 1]$$

$$1 + 1 + 1 + 1 + 1 + \cdots$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

Ratio Test. If, as the number of terms approaches infinity as its limit, the ratio of the $(n+1)$th term to the nth term approaches some finite limit (a), the series is convergent if (a) is less than 1, divergent if (a) is greater than 1, and indeterminate by this method if $(a) = 1$.

Oscillating Series. A series whose terms are alternately positive and negative is convergent if each term is numerically less than the preceding term.

Standard Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

$$e^{jx} = e^{\sqrt{-1}\,x} = 1 + jx - \frac{x^2}{2!} - \frac{jx^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-jx} = e^{-\sqrt{-1}\,x} = 1 - jx - \frac{x^2}{2!} + \frac{jx^3}{3!} + \frac{x^4}{4!} - \cdots$$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = \lim_{n \doteq \infty} \left(1 + \frac{1}{n}\right)^n$$
$$= 2.7182818 \ldots$$

$$a^x = 1 + x \log a + \frac{(x \log a)^2}{2!} + \frac{(x \log a)^3}{3!} + \cdots$$

$$\log x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right] \quad [x>0]$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad [1 \geqq x > -1]$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \quad [1 > x \geqq -1]$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2\,x^5}{15} + \frac{17\,x^7}{315} + \frac{62\,x^9}{2835} + \cdots \qquad \left[\frac{\pi}{2} > x > -\frac{\pi}{2}\right]$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2\,x^5}{945} - \frac{x^7}{4725} - \cdots \qquad [x^2 < \pi^2]$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \cdots \quad [1 > x > -1]$$

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad [1 > x > -1]$$

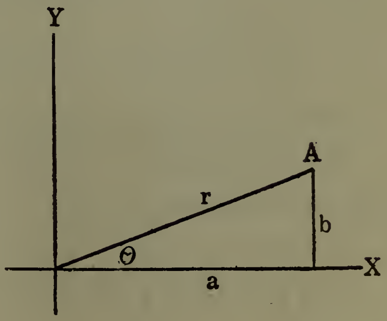
$$\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3\,x^3} - \frac{1}{5\,x^5} + \cdots \qquad [x^2 > 1]$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

Complex Imaginary Quantities

The imaginary unit $= \sqrt{-1} = j$



In representing complex imaginary quantities, it is usual to represent real quantities in the direction of the horizontal or X-axis, and imaginaries in the direction of the vertical or Y-axis. Multipli-

cation by the imaginary unit, j, revolves a quantity through 90 degrees, in counter-clockwise direction.

A **complex number** is the sum of a real and an imaginary, thus:

$$A = a + jb = a + \sqrt{-1}\, b$$

is a **complex number.**

A **complex number** may be written in any of the following identical forms:

$$A = a + jb = r(\cos\theta + j\sin\theta) = re^{j\theta} \quad [\theta \text{ in radians}]$$

in which $\begin{cases} a = r\cos\theta, \\ b = r\sin\theta. \end{cases}$

The **magnitude** of the complex number, $a + jb$, is

$$r = \sqrt{a^2 + b^2}$$

Addition and Subtraction of complex quantities:

To add two complex quantities, combine the real parts, and then the imaginaries, thus:

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

In the same way, to subtract two complex quantities:

$$(a + jb) - (c + jd) = (a - c) + j(b - d)$$

Multiplication of complex quantities:

To find the product of two complex numbers, multiply out as in ordinary algebra, remembering that $j^2 = -1$, thus:

$$(a + jb)(c + jd) = (ac - bd) + j(ad + bc)$$

Division of complex quantities:

To divide two complex quantities, rationalize the denominator as follows:

$$\frac{a + jb}{c + jd} = \frac{a + jb}{c + jd} \times \frac{c - jd}{c - jd} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}$$

Exponential Transformations

$$e^{jax} = \cos ax + j \sin ax$$
$$e^{-jax} = \cos ax - j \sin ax$$
$$\sin ax = \frac{e^{jax} - e^{-jax}}{2j}$$
$$\cos ax = \frac{e^{jax} + e^{-jax}}{2}$$

(e is the base of the hyperbolic logarithms; j equals $\sqrt{-1}$).

De Moivre's Theorem:

$$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta$$

Permutations and Combinations

The number of **permutations** of n different things taken r at a time is

$$P_r = n(n-1)\ .\ .\ .\ (n-r+1) = \frac{n!}{(n-r)!}$$

For n different things taken all at a time, the number of **permutations** is

$$P_n = n(n-1)\ .\ .\ .\ (2)(1) = n!$$

The number of **permutations** of n things taken all at a time, n_1 being alike, n_2 alike, n_3 alike, etc., is

$$P = \frac{n!}{n_1!\ n_2!\ n_3!\ .\ .\ .}$$

The number of **combinations** of n things taken r at a time is

$$C_r = \frac{n(n-1)\ .\ .\ .\ (n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$$

For n things taken 1, 2, 3, . . . n at a time, the total number of **combinations** is

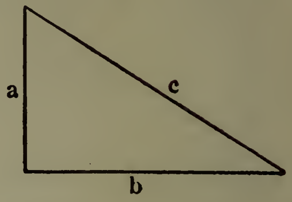
$$C = 2^n - 1$$

GEOMETRY

Plane Figures

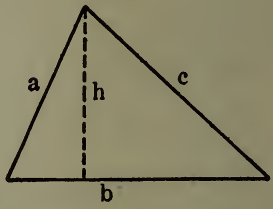
Right Triangle

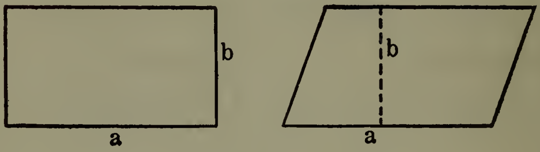
$$c = \sqrt{a^2 + b^2}$$
$$a = \sqrt{c^2 - b^2}$$
$$b = \sqrt{c^2 - a^2}$$
$$\text{area} = \tfrac{1}{2}\, ab$$



Any Triangle

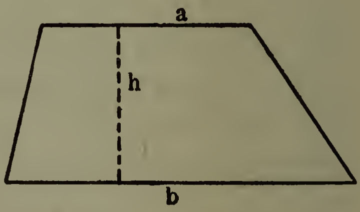
$$\text{area} = \tfrac{1}{2}\, bh$$
$$\text{area} = \sqrt{s(s-a)\,(s-b)\,(s-c)}$$
$$s = \tfrac{1}{2}\,(a + b + c)$$





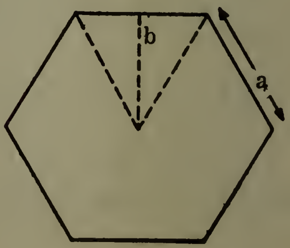
Parallelogram

$$\text{area} = ab$$



Trapezoid

$$\text{area} = \tfrac{1}{2}\, h\,(a + b)$$

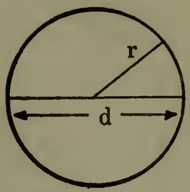


Regular Polygon

$$\text{area} = \tfrac{1}{2}\, abn$$
$$n = \text{number of sides}$$

Circle

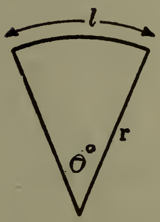
$$\text{circumference} = 2\,\pi r$$
$$= \pi d$$
$$\text{area} = \pi r^2$$
$$= \pi \frac{d^2}{4}$$



Sector of Circle

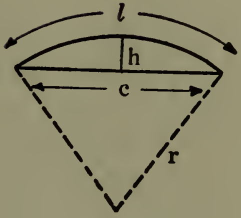
$$\text{arc} = l = \pi r \frac{\theta^\circ}{180^\circ}$$

$$\text{area} = \tfrac{1}{2}\,rl = \pi r^2 \frac{\theta^\circ}{360^\circ}$$



Segment of Circle

$$\text{chord} = c = 2\,\sqrt{2\,hr - h^2}$$
$$\text{area} = \tfrac{1}{2}\,rl - \tfrac{1}{2}\,c\,(r - h)$$

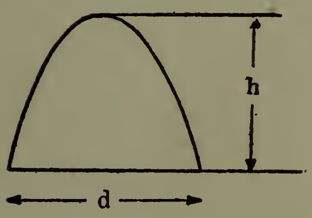


Parabola

$$\text{length of arc} = \frac{d^2}{8\,h}\left[\sqrt{c\,(1+c)} + 2.0326\,\log_{10}\left(\sqrt{c} + \sqrt{1+c}\right)\right]$$

in which

$$c = \left(\frac{4\,h}{d}\right)^2$$
$$\text{area} = \tfrac{2}{3}\,dh$$



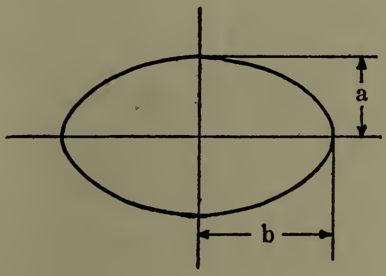
Ellipse

circumference =

$$\pi\,(a+b)\,\frac{64 - 3\left(\dfrac{b-a}{b+a}\right)^4}{64 - 16\left(\dfrac{b-a}{b+a}\right)^2}$$

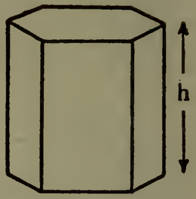
(close approximation)

$$\text{area} = \pi ab$$



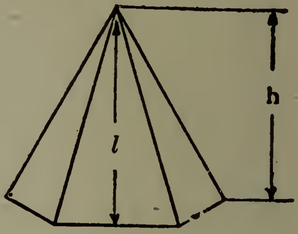
Solids

Right Prism



lateral surface = perimeter of base $\times h$
volume = area of base $\times h$

Pyramid



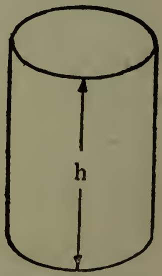
lateral area $= \frac{1}{2}$ perimeter of base $\times l$

volume = area of base $\times \frac{h}{3}$

Frustum of Pyramid

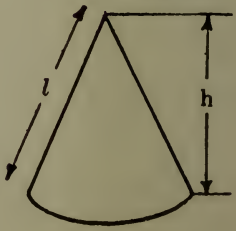
lateral surface $= \frac{1}{2} l (P + p)$
P = perimeter of lower base
p = perimeter of upper base
volume $= \frac{1}{3} h \left[A + a + \sqrt{Aa}\right]$
A = area of lower base
a = area of upper base

Right Circular Cylinder



lateral surface $= 2\pi rh$
r = radius of base
volume $= \pi r^2 h$

Right Circular Cone



lateral surface $= \pi rl$
r = radius of base
volume $= \frac{1}{3}\pi r^2 h$

Frustum of Right Circular Cone

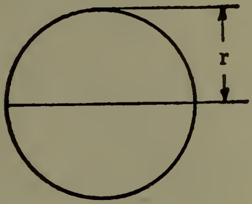
lateral surface $= \pi l\ (R + r)$

R = radius of lower base

r = radius of upper base

volume $= \frac{1}{3}\pi h\ [R^2 + Rr + r^2]$

Sphere



surface $= 4\pi r^2$

volume $= \frac{4}{3}\pi r^3$

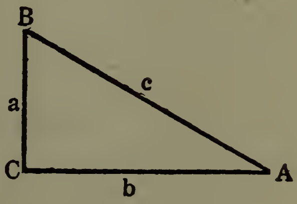
Segment of Sphere

volume of segment

$= \frac{1}{6}\,a\pi\ [3\ (r_1^2 + r_2^2) + a^2]$

PLANE TRIGONOMETRY

Right Triangle



$\sin A = \frac{a}{c}$ $\qquad$ $\cos A = \frac{b}{c}$

$\tan A = \frac{a}{b}$ $\qquad$ $\cot A = \frac{b}{a}$

$\sec A = \frac{c}{b}$ $\qquad$ $\operatorname{cosec} A = \frac{c}{a}$

$$\sin A = \cos\left(\frac{\pi}{2} - A\right) = -\cos\left(\frac{\pi}{2} + A\right)$$

$$\cos A = \sin\left(\frac{\pi}{2} - A\right) = \sin\left(\frac{\pi}{2} + A\right)$$

$$\tan A = \cot\left(\frac{\pi}{2} - A\right) = -\cot\left(\frac{\pi}{2} + A\right)$$

$$\cot A = \tan\left(\frac{\pi}{2} - A\right) = -\tan\left(\frac{\pi}{2} + A\right)$$

$$\sec A = \operatorname{cosec}\left(\frac{\pi}{2} - A\right) = \operatorname{cosec}\left(\frac{\pi}{2} + A\right)$$

$$\operatorname{cosec} A = \sec\left(\frac{\pi}{2} - A\right) = -\sec\left(\frac{\pi}{2} + A\right)$$

$\sin(-A) = -\sin A$ $\quad$ $\cos(-A) = \cos A$

$\tan(-A) = -\tan A$ $\quad$ $\cot(-A) = -\cot A$

$\sec(-A) = \sec A$ $\quad$ $\operatorname{cosec}(-A) = -\operatorname{cosec} A$

NUMERICAL VALUES

Angle . .	0°	30°	45°	60°	90°
sin.	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos.	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan.	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞
cot.	∞	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Trigonometric Formulæ

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x} \qquad \operatorname{cosec} x = \frac{1}{\sin x}$$

$$\tan x = \frac{1}{\cot x} \qquad \cot x = \frac{1}{\tan x}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\sec^2 x = 1 + \tan^2 x$$

$$\operatorname{cosec}^2 x = 1 + \cot^2 x$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\cot(x + y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$\cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$$

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}$$

$$\sin \tfrac{1}{2} x = \sqrt{\frac{1 - \cos x}{2}}$$

$$\cos \tfrac{1}{2} x = \sqrt{\frac{1 + \cos x}{2}}$$

$$\tan \tfrac{1}{2} x = \frac{1 - \cos x}{\sin x}$$

$$\sin x + \sin y = 2 \sin \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y)$$

$$\sin x - \sin y = 2 \cos \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y)$$

$$\cos x + \cos y = 2 \cos \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y)$$

$$\cos x - \cos y = -2 \sin \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y)$$

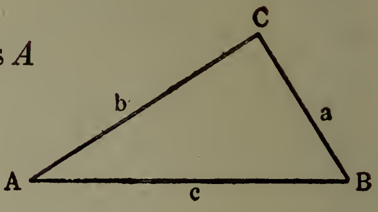
Solution of Any Plane Triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2\,bc\cos A$$

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$$

$$\tan A = \frac{a\sin C}{b - a\cos C}$$



SPHERICAL TRIGONOMETRY

Right Spherical Triangles

$\cos c = \cos a \cos b$

$\sin a = \sin c \sin A$

$\sin b = \sin c \sin B$

$\cos A = \cos a \sin B$

$\cos B = \cos b \sin A$

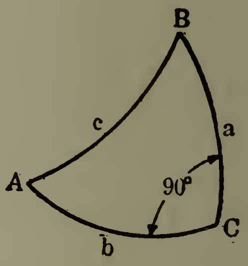
$\cos A = \tan b \cot c$

$\cos B = \tan a \cot c$

$\sin b = \tan a \cot A$

$\sin a = \tan b \cot B$

$\cos c = \cot A \cot B$



Oblique Spherical Triangles

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

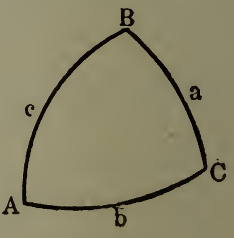
$\cos a = \cos b \cos c + \sin b \sin c \cos A$

$\cos A = \sin B \sin C \cos a - \cos B \cos C$

$\cot a \sin b = \cot A \sin C + \cos C \cos b$

$s = \frac{1}{2}(a + b + c)$

$S = \frac{1}{2}(A + B + C)$



$$\sin\left(\frac{A}{2}\right) = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}$$

$$\cos\left(\frac{A}{2}\right) = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}$$

$$\tan\left(\frac{A}{2}\right) = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}$$

$$\sin\left(\frac{a}{2}\right) = \sqrt{-\frac{\cos S \cos(S-A)}{\sin B \sin C}}$$

$$\cos\left(\frac{a}{2}\right) = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B \sin C}}$$

$$\tan\left(\frac{a}{2}\right) = \sqrt{-\frac{\cos S \cos(S-A)}{\cos(S-B)\cos(S-C)}}$$

$$\tan\tfrac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c$$

$$\tan\tfrac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c$$

$$\tan\tfrac{1}{2}(A-B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C$$

$$\tan\tfrac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C$$

$$\tan\tfrac{1}{2}c = \frac{\sin\frac{1}{2}(A+B)\tan\frac{1}{2}(a-b)}{\sin\frac{1}{2}(A-B)}$$

Application of Spherical Trigonometry to Navigation

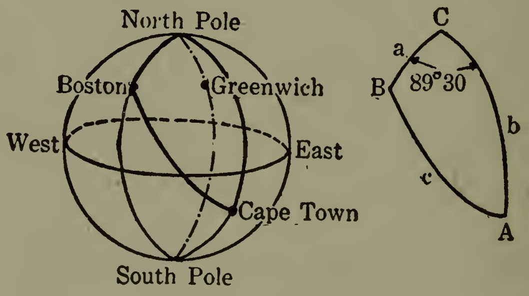
To find the shortest distance between two points on the earth's surface and the bearing of each from the other, the latitude and longitude of each being given. (From W. A. Granville's "Plane and Spherical Trigonometry.")

(1) Subtract the latitude of each place algebraically from 90°, taking North latitudes as positive and South latitudes as negative. The results will be the two sides of a spherical triangle.

(2) Find the difference of longitude of the two places by subtracting the lesser longitude from the greater if both are East or both are West; but adding the two if one is East and the other West. This gives the included angle of the triangle. If the difference of longitude found is greater than 180°, then subtract it from 360° and use the remainder as the included angle.

(3) Solving the triangle by the formulæ for $\tan \frac{1}{2}(A - B)$, $\tan \frac{1}{2}(A + B)$, and $\tan \frac{1}{2} c$, the third side gives the shortest distance between the two points in degrees of arc, and the angles give the bearings. The number of minutes in the arc will be the distance between the places in nautical miles.

Illustration. Find the shortest distance along the earth's surface between Boston (latitude 42° 21′ N.,



longitude 71° 4′ W.) and Capetown (latitude 33° 56′ S., longitude 18° 26′ E.) and the bearing of each city from the other.

(1) $a = 90° - 42° \, 21' = 47° \, 39'$

$b = 90° - (-33° \, 56') = 123° \, 56'$

(2) $C = 71° 4' + 18° 26' = 89° 30'$ = difference in longitude.

(3) Solving the triangle as explained above, we get

$c = 68° 14' = 68.23° = 4094$ nautical miles.

$A = 52° 43'$ = bearing of Boston from Capetown.

$B = 116° 43'$ = bearing of Capetown from Boston.

PLANE ANALYTIC GEOMETRY

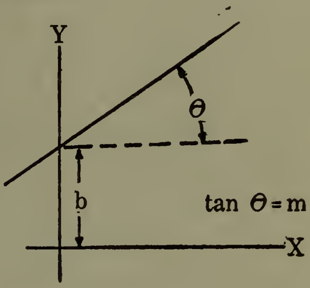
The Straight Line

I. The slope equation:

$y = mx + b$

m = slope = $\tan\theta$

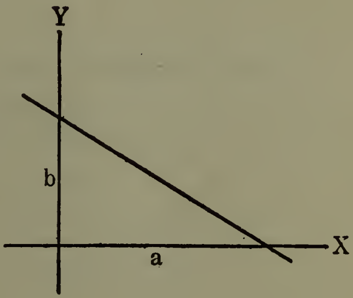
b = intercept on Y-axis



II. The intercept equation:

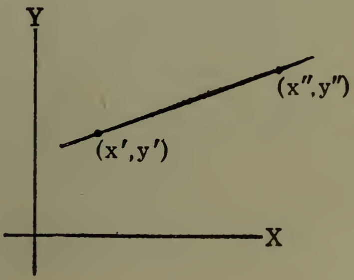
$$\frac{x}{a} + \frac{y}{b} = 1$$

where a and b are the intercepts on the X and Y-axes.



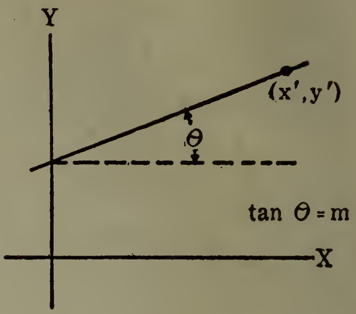
III. Line through the points (x', y') and (x'', y''):

$$\frac{y - y'}{y'' - y'} = \frac{x - x'}{x'' - x'}$$



IV. **Line** through the point (x', y'), with slope m:

$$y - y' = m(x - x')$$



V. **Distance from** the point (x', y') to the line $Ax + By + C = 0$:

$$d = \frac{Ax' + By' + C}{\pm\sqrt{A^2 + B^2}}$$

VI. **Distance between** the points (x', y') and (x'', y''):

$$d = \sqrt{(x' - x'')^2 + (y' - y'')^2}$$

Transformation from Rectangular to Polar Coördinates

$$x = r\cos\theta$$

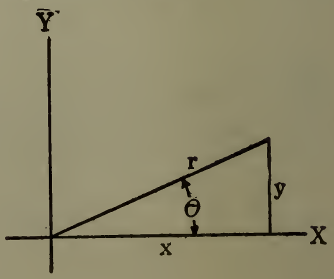
$$y = r\sin\theta$$

$$r = \text{radius vector} = \sqrt{x^2 + y^2}$$

$$\theta = \text{polar angle} = \tan^{-1}\frac{y}{x}$$

$$\sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}$$

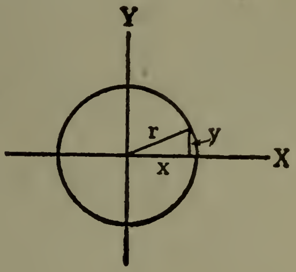
$$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}$$



The Circle

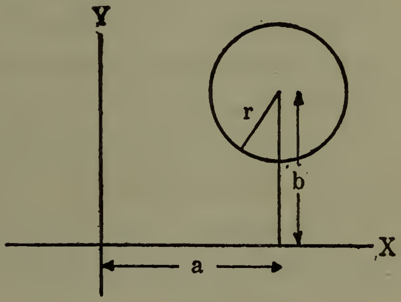
I. Circle of radius r with **center at origin:**

$$x^2 + y^2 = r^2$$



II. Circle of radius r with its **center at the point** (a, b):

$$(x - a)^2 + (y - b)^2 = r^2$$



III. **Tangent** at the point (a, b) of the circle $x^2 + y^2 = r^2$ is

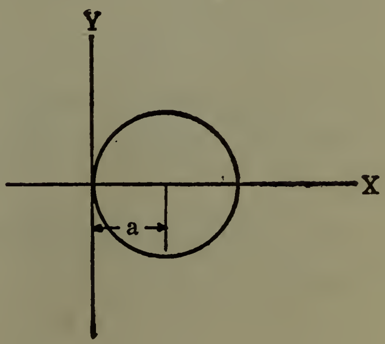
$$ax + by = r^2$$

IV. Slope equation of the **tangent** to the circle $x^2 + y^2 = r^2$ is

$$y = mx \pm r\sqrt{m^2 + 1}$$

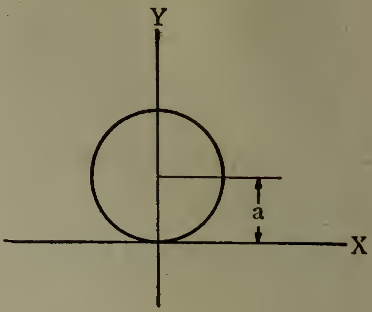
V. **Polar equation** of circle of radius a passing through the origin, and having its center on the X-axis:

$$r = 2\,a \cos\theta$$



VI. **Polar equation** of circle of radius a passing through the origin, and having its center on the Y-axis:

$$r = 2\,a \sin\theta$$



Parabola

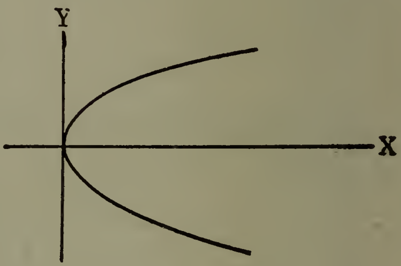
Definition. The **parabola** is the curve generated by a point moving so as to remain always equidistant from a given fixed point and a given fixed line.

The fixed point is called the **focus**; the fixed line is called the **directrix.**

I. **Parabola** with its axis along the X-axis and vertex at origin:

$$y^2 = 4\,ax$$

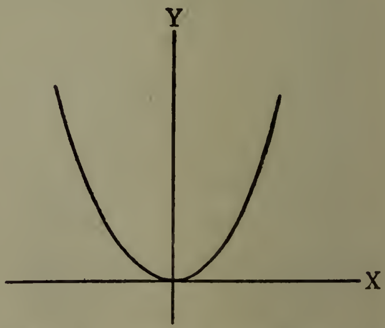
where a is the distance from the origin to the focus.



II. **Parabola** having its axis along the Y-axis and vertex at origin:

$$x^2 = 4\,ay$$

where a is the distance from the origin to the focus.



III. **General equation** of a parabola with axis parallel to the X-axis:

$$x = ay^2 + by + c$$

the **vertex** is at the point

$$\left(-\frac{b^2 - 4\,ac}{4\,a}, -\frac{b}{2\,a}\right)$$

IV. **General equation** of a parabola with axis parallel to the *Y*-axis:

$$y = ax^2 + bx + c$$

the **vertex** is at the point

$$\left(-\frac{b}{2\,a}, -\frac{b^2 - 4\,ac}{4\,a}\right)$$

V. Slope equation of the **tangent** to the parabola $y^2 = 4\,ax$ is

$$y = mx + \frac{a}{m}$$

VI. Slope equation of the **tangent** to the parabola $x^2 = 4\,ay$ is

$$y = mx - am^2$$

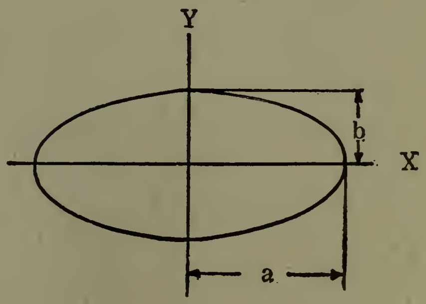
Ellipse

Definition. The **ellipse** is the curve generated by a point moving so that the sum of its distances from two fixed points is always constant. The fixed points are called the **foci.**

I. Equation of ellipse with **center at origin:**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are one-half the major and minor axes.



II. Slope equation of the **tangent** to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

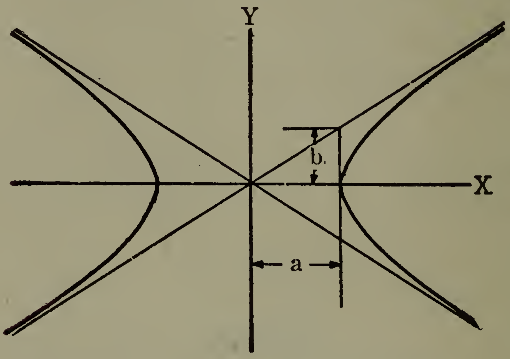
$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

Hyperbola

Definition. The **hyperbola** is the curve generated by a point moving so that the difference of its distances from two fixed points is always constant.

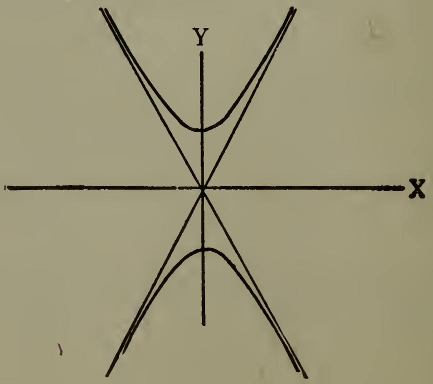
I. Equation of hyperbola with **center at origin:**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



II. Equation of **conjugate hyperbola:**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$



III. Equations of **asymptotes** of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are

$$y = \frac{b}{a}x \qquad y = -\frac{b}{a}x$$

IV. Slope equation of the **tangent** to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

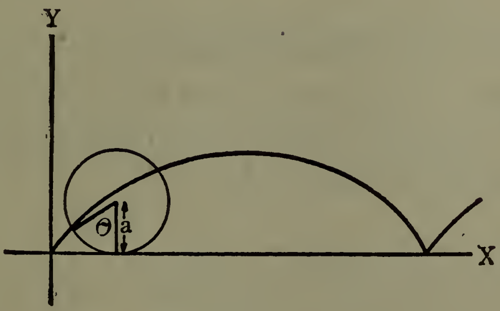
$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

V. Slope equation of the **tangent** to the conjugate hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is

$$y = mx \pm \sqrt{b^2 - a^2m^2}$$

Cycloid

Definition. The **cycloid** is the curve generated by a point on the circumference of a circle as the circle rolls along a straight line.

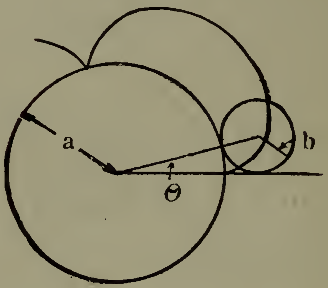


$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta)$$
$$x = a \operatorname{vers}^{-1}\frac{y}{a} - \sqrt{2ay - y^2}$$

where a is the radius of the rolling circle.

Epicycloid

Definition. The **epicycloid** is the curve generated by a fixed point on the circumference of a circle which rolls **externally** on the circumference of a fixed circle.



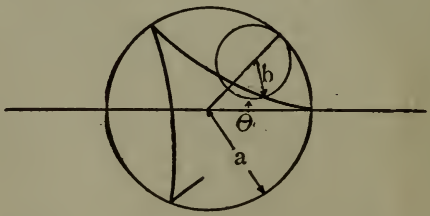
$$x = (a + b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right)$$

$$y = (a + b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right)$$

where a is the radius of the fixed circle, and b the radius of the rolling circle.

Hypocycloid

Definition. The **hypocycloid** is the curve generated by a point on a circle which rolls **internally** along the circumference of a fixed circle.



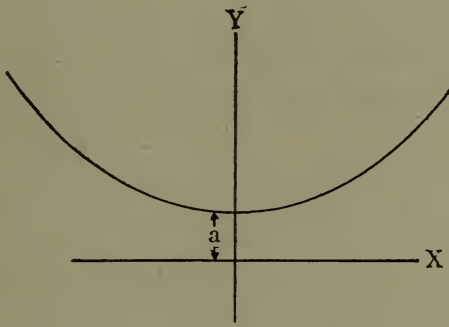
$$x = (a - b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right)$$

$$y = (a - b) \sin \theta - b \sin \left(\frac{a - b}{b} \theta \right)$$

where a is the radius of the fixed circle and b the radius of the rolling circle.

The Catenary

The **catenary** is the curve which a heavy cord or perfectly flexible chain of uniform density forms, due



to its own weight, when freely suspended between two points.

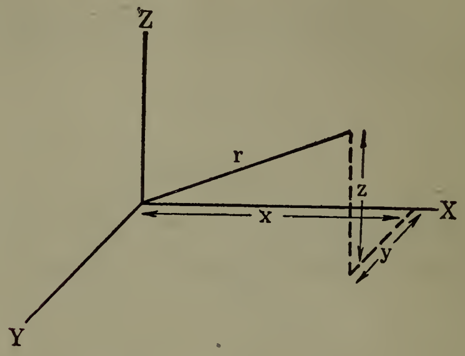
$$y = \frac{a}{2} \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right) = a \cosh \frac{x}{a}$$

SOLID ANALYTIC GEOMETRY

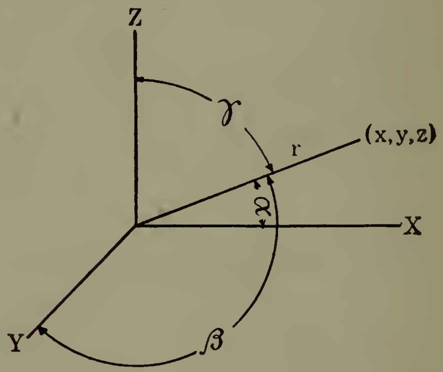
The **direction cosines** of a line in space passing through the origin are the cosines of the angles which the line makes with the rectangular coördinate axes. The direction cosines of **any line** in space are the direction cosines of a line parallel to it and passing through the origin.

I. **Distance from** the point (x, y, z) to the origin:

$$r = \sqrt{x^2 + y^2 + z^2}$$



II. The **direction cosines** of the line from the point (x, y, z) to the origin are:



$$\cos \alpha = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\cos \beta = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\cos \gamma = \frac{z}{r} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

III. The **sum of the squares** of the direction cosines of a line is equal to 1,

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

IV. **Distance between** the points (x, y, z) and (x', y', z'):

$$d = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

V. Direction cosines of a **line joining the points** (x, y, z) and (x', y', z'):

$$\cos\alpha = \frac{x - x'}{d} = \frac{x - x'}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$

$$\cos\beta = \frac{y - y'}{d} = \frac{y - y'}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$

$$\cos\gamma = \frac{z - z'}{d} = \frac{z - z'}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$

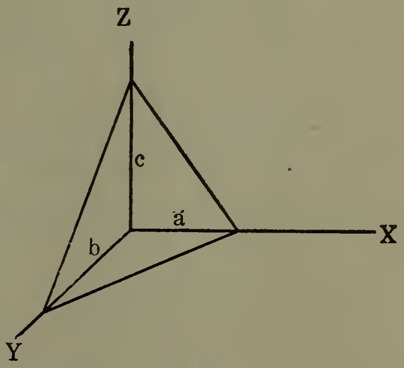
VI. The **angle between** two lines in terms of their direction cosines:

$$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'$$

VII. **Intercept equation** of a plane:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where a, b, and c are the intercepts of the plane on the X, Y, and Z axes.



VIII. General equation of a **plane**:

$$Ax + By + Cz + D = 0$$

IX. **Distance from** the point (x', y', z') to the plane $Ax + By + Cz + D = 0$:

$$d = \frac{Ax' + By' + Cz' + D}{\pm\sqrt{A^2 + B^2 + C^2}}$$

X. **Straight line** through the two points (x'', y'', z'') and (x', y', z'):

$$\frac{x - x'}{x'' - x'} = \frac{y - y'}{y'' - y'} = \frac{z - z'}{z'' - z'}$$

XI. **Straight line** through the point (x', y', z'), and making the angles α, β, and γ with the coördinate axes:

$$\frac{x - x'}{\cos\alpha} = \frac{y - y'}{\cos\beta} = \frac{z - z'}{\cos\gamma}$$

XII. General equation of a **straight line** is given by the equations of two intersecting planes:

$$A'x + B'y + C'z + D' = 0$$
$$A''x + B''y + C''z + D'' = 0$$

CALCULUS

Application of Differential Calculus

The following list includes some of the principal formulæ necessary for the solution of geometrical and physical problems, relating to any curve $y = f(x)$.

Rectangular Coördinates:

Slope of the tangent at the point $(x, y) = \dfrac{dy}{dx}$

Slope of the normal $= -\dfrac{dx}{dy}$

Equation of the tangent at the point (x_o, y_o), x_o and y_o being the coördinates of the given point, is

$$y_o - y = \frac{dy_o}{dx_o}(x_o - x)$$

Equation of the normal at (x_o, y_o) is

$$(y_o - y) = -\frac{dx_o}{dy_o}(x_o - x)$$

The intercept of the tangent on the X-axis is $x - y\frac{dx}{dy}$

The intercept of the tangent on the Y-axis is $y - x\frac{dy}{dx}$

The intercept of the normal on the X-axis is $x + y\frac{dy}{dx}$

The intercept of the normal on the Y-axis is $y + x\frac{dx}{dy}$

Length of the tangent from its point of contact with the curve to the X-axis is

$$y\sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

Length of the tangent from its point of contact with the curve to the Y-axis is

$$x\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Length of the normal from its point of contact with the curve to the X-axis is

$$y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Length of the normal from its point of contact with the curve to the Y-axis is

$$x\sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

Length of the subtangent $= y\frac{dx}{dy}$

Length of the subnormal $= y\frac{dy}{dx}$

Differential length of the arc $= ds = \sqrt{(dx)^2 + (dy)^2}$

$$= dy\sqrt{1+\left(\frac{dx}{dy}\right)^2} = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\text{Radius of curvature} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

Curvature is the reciprocal of radius of curvature.

Length of the perpendicular from the origin on the tangent (to the curve) is

$$\frac{x\frac{dy}{dx} - y}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}.$$

Polar Coördinates:

$\tan \psi = r \dfrac{d\theta}{dr}$, where ψ is the angle between the radius vector and that part of the tangent to the curve at (r, θ) drawn back toward the initial line.

$$\text{Length of polar subtangent} = r^2 \frac{d\theta}{dr}$$

$$\text{Length of polar subnormal} = \frac{dr}{d\theta}$$

$$\text{Differential length of arc} = ds = \sqrt{(dr)^2 + r^2\,(d\theta)^2}$$

$$= dr\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} = d\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

Length of the perpendicular from the pole on the tangent $= p = r^2 \dfrac{d\theta}{ds}$, also,

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

Formulæ of Differential Calculus

$$d\,(au) = a\,du$$
$$d\,(u+v) = du + dv$$
$$d\,(uv) = v\,du + u\,dv$$
$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$
$$d\,(x^n) = nx^{n-1}\,dx$$
$$d\,(x^y) = yx^{y-1}\,dx + x^y \log_e x\,dy$$
$$d\,(e^x) = e^x\,dx$$
$$d\,(a^u) = a^u \log_e a\,du$$
$$d\,(\log_e x) = \frac{1}{x}\,dx$$
$$d\,(\sin x) = \cos x\,dx$$
$$d\,(\cos x) = -\sin x\,dx$$
$$d\,(\tan x) = \sec^2 x\,dx$$
$$d\,(\cot x) = -\operatorname{cosec}^2 x\,dx$$
$$d\,(\sec x) = \sec x \tan x\,dx$$
$$d\,(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x\,dx$$
$$d\,(\sin^{-1} x) = \frac{dx}{\sqrt{1-x^2}}$$
$$d\,(\cos^{-1} x) = -\frac{dx}{\sqrt{1-x^2}}$$
$$d\,(\tan^{-1} x) = \frac{dx}{1+x^2}$$
$$d\,(\cot^{-1} x) = -\frac{dx}{1+x^2}$$
$$d\,(\sec^{-1} x) = \frac{dx}{x\sqrt{x^2-1}}$$
$$d\,(\operatorname{cosec}^{-1} x) = -\frac{dx}{x\sqrt{x^2-1}}$$

Maxima and Minima

The **maximum** or **minimum** values of a given function $y = f(x)$ are obtained as follows:

(1) Find the first derivative $\frac{dy}{dx}$ and equate it to zero.

(2) Solve the resulting equation for values of x.

(3) In order to determine whether these values of x make y maximum or minimum, obtain the second derivative $\frac{d^2y}{dx^2}$ of the given function.

(4) Substitute separately in the expression for $\frac{d^2y}{dx^2}$ each of the values of x found above. Values of x that make $\frac{d^2y}{dx^2}$ positive correspond to minimum values of the function, and values of x that make $\frac{d^2y}{dx^2}$ negative correspond to maximum values of the function.

(5) Substituting these values of x in the given function $y = f(x)$, we obtain the maximum or minimum values of y.

Illustrative Example. Find the values of x which will make the function $y = 6x + 3x^2 - 4x^3$ a maximum or a minimum, and find the corresponding values of the function y.

(1) The first derivative of y is

$$\frac{dy}{dx} = 6 + 6x - 12x^2$$

(2) The values of x which make y maximum or minimum will make $\frac{dy}{dx} = 0$; therefore

$$6 + 6x - 12x^2 = 0, \quad \text{or} \quad x^2 - \tfrac{1}{2}x = \tfrac{1}{2}$$

solving, $x = \frac{1}{4} \pm \frac{3}{4} = +1$ or $-\frac{1}{2}$

Hence, the maximum or minimum values of y must occur when $x = 1$ or $-\frac{1}{2}$.

(3) To determine whether these values are maxima or minima, we obtain the second derivative of y; thus:

$$\frac{d^2y}{dx^2} = 6 - 24x$$

(4) When $x = 1$, $\frac{d^2y}{dx^2} = -18$, which corresponds to a maximum value of y.

When $x = -\frac{1}{2}$, $\frac{d^2y}{dx^2} = +18$, which corresponds to a minimum value of y.

(5) Substituting these values of x in the given function, we have

when $x = 1$, $y = 6 + 3 - 4 = 5$, a maximum

when $x = -\frac{1}{2}$, $y = -3 + \frac{3}{4} + \frac{1}{2} = -\frac{7}{4}$, a minimum

Taylor's and Maclaurin's Series

Taylor's Series:

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + \cdots$$

where f denotes the function, f' the first derivative, f'' the second derivative, etc.

Maclaurin's Series:

$$f(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots$$

where $f(0)$ denotes the value of the function when 0 is substituted for x, $f'(0)$ the value of the first derivative when 0 is substituted for x, etc.

APPLICATION OF INTEGRAL CALCULUS

Lengths of Curves

Rectangular Coördinates:

$$\text{length of curve} = s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

From the equation of the given curve, find y in terms of x; then differentiate in order to obtain $\frac{dy}{dx}$, and substitute its value in the formula. The lower limit a is the initial value of x, and the upper limit b the final value of x.

Or, similarly, by solving for x in terms of y, and obtaining $\frac{dx}{dy}$, the length of the curve is given by the formula

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

where c and d are the initial and final values of y.

Polar Coördinates:

$$\text{length of curve} = s = \int_a^b \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$$

where a and b are the limiting values of r.

Or,

$$\text{length of curve} = s = \int_{\theta'}^{\theta''} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

where θ' and θ'' are the limiting values of θ.

Plane Areas

Rectangular Coördinates:

The area included between a curve, the X-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{area} = A = \int_a^b y\,dx$$

The value of y in terms of x is found from the given equation and substituted in the formula. The initial value of x is a, and the final value b.

Similarly, the area included between a curve, the Y-axis, and the horizontal lines $y = c$ and $y = d$ is

$$\text{area} = A = \int_c^d x\,dy$$

where c and d are the limits of y.

Polar Coördinates:

The area included between a given curve and two given radii is

$$\text{area} = A = \tfrac{1}{2}\int_{\theta'}^{\theta''} r^2\,d\theta$$

where θ'' and θ' are the limiting values of θ.

Areas of Surfaces of Revolution

For revolution about the X-axis,

$$\text{area} = A = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

where the value of $\left(\frac{dy}{dx}\right)$ is found from the given equation. The initial value of x is a, and the final value b.

For revolution about the Y-axis,

$$\text{area} = A = 2\pi \int_c^d x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

where c and d are the limiting values of y.

Volumes of Solids of Revolution

Rectangular Coördinates:

$$\text{volume} = V_x = \pi \int_a^b y^2\, dx$$

is the formula for the volume generated by revolving the given curve about the X-axis. The limiting values of x are a and b.

Similarly, the volume generated by revolving the plane figure about the Y-axis equals

$$V_y = \pi \int_c^d x^2\, dy$$

where c and d are the initial and final values of y.

Polar Coördinates:

When the plane figure is revolved about the X-axis, the volume generated is

$$V_x = 2\pi \int\int r^2 \sin\theta\, d\theta\, dr$$

For revolution about the Y-axis, the volume generated is

$$V_y = 2\pi \int\int r^2 \cos\theta\, d\theta\, dr$$

INDETERMINATE FORMS

If the fraction $\frac{f(x)}{F(x)}$ gives rise to the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, when x approaches a as a limit, the in-

determinate form may be replaced by a new fraction, $\frac{f'(x)}{F'(x)}$, the numerator of which is equal to the derivative of the given numerator, and the new denominator is equal to the derivative of the given denominator. The value of this new fraction, as x approaches a, is the limiting value of the given fraction. If this again becomes indeterminate, it may be necessary to repeat the process several times.

Example. Find the limiting value, when $x = 1$, of the fraction

$$\frac{x^2 + x - 2}{x^2 - 1}$$

$$\frac{f(x)}{F(x)} = \frac{x^2 + x - 2}{x^2 - 1} = \frac{0}{0}, \text{ when } x = 1$$

$$\frac{f'(x)}{F'(x)} = \frac{2x + 1}{2x} = \frac{3}{2}, \text{ when } x = 1$$

Hence, the required limiting value is $\frac{3}{2}$.

SOLUTION OF EQUATIONS

Algebraic equations may be solved by Newton's method of approximation. Thus, let it be required to solve an equation of the form $Ax^3 + Bx^2 + Cx = D$. Find, by trial, a number, r, nearly equal to the root sought, and let $r + h$ denote the exact value of the root, where h is a small quantity the value of which must be determined. Substituting $r + h$ for x in the given equation and neglecting all powers of h higher than the first, we have, approximately,

$$h = \frac{Ar^3 + Br^2 + Cr - D}{-3Ar^2 - 2Br - C}$$

It will be observed that the numerator of the above

fraction is the first member of the given equation after D has been transposed and x changed to r, and the denominator is the **first derivative** of the numerator with its sign reversed. The correction h added, with its proper sign, to the assumed root r, gives a closer approximation to the value of x. Repeat the operation with the corrected value of r, and a second correction will be obtained which will give a nearer value of the root; two corrections generally give sufficient accuracy.

Illustration. Find a root of the equation

$$x^3 + 2x^2 + 3x = 50$$

The value of h is

$$h = \frac{r^3 + 2r^2 + 3r - 50}{-3r^2 - 4r - 3}$$

By trial, we find that x is nearly equal to 3. On substituting 3 for r, we have

$$h = -\frac{2}{21} = -0.1, \text{ approximately}$$

Hence, $x = 2.9$, nearly. If we substitute this new value of r, the new value of h equals $+0.00228$. Hence $x = 2.90228$. If we repeat the operation with this last value of r, the value of h is then found to be $+0.0000034$. Hence $x = 2.9022834$.

CURVE TRACING

The usual method of tracing curves consists in assigning a series of different values to one of the variables, and calculating the corresponding series of values of the other, thus determining a definite number of points on the curve. By drawing a curve through

these points, we obtain a graphical representation of the given equation.

The **general form** and **peculiarities** of the curve can be easily determined and sketched by the following steps:

(1) If possible, solve the equation of the given curve for one of its variables, y for example. If the equation then contains only even powers of x, it is symmetrical with the Y-axis.

Or if, when solved for x, it contains only even powers of y, it is symmetrical with the X-axis.

(2) Find the points in which the curve cuts the axes by solving the equation of the given curve in turn with the equations $x = 0$ and $y = 0$.

(3) Find the values of x, if any, which make y infinite; similarly, test for infinite values of x.

(4) Find the value of the first derivative $\frac{dy}{dx}$; and thence deduce the maximum and minimum points of the curve.

In **tracing polar curves,** write the equation, if possible, in the form $r = f(\theta)$; and give θ such values as make r easily found, as for example, $0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi$, etc. Putting $\frac{dr}{d\theta} = 0$, we find the values of θ for which r is a maximum or minimum.

METHODS OF INTEGRATION

(By **parts, substitution,** etc.)

When the numerator of a fraction contains a variable to an **equal** or a **higher** power than the denominator, the fraction must be reduced to a mixed quantity (by

actually dividing the denominator into the numerator) before it can be integrated.

If an expression cannot be integrated by the formulæ given in the table of integrals, one of the following methods may be used to obtain a solution.

Partial Fractions

A fraction may be resolved into partial fractions, which can be integrated separately.

Example. To integrate

$$\frac{1}{(x+a)(x+b)}dx$$

Let

$$\frac{1}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

where we must determine A and B.

Clearing of fractions,

$$1 = A(x+b) + B(x+a) = (A+B)x + (bA+aB)$$

The coefficients of like powers of x on both sides of the equation are equal; therefore,

$$A + B = 0$$
$$bA + aB = 1$$

whence $\quad A = \dfrac{1}{b-a} \quad$ and $\quad B = \dfrac{1}{a-b}$

and

$$\int \frac{1}{(x+a)(x+b)}dx = \int \frac{\left(\frac{1}{b-a}\right)}{(x+a)}dx + \int \frac{\left(\frac{1}{a-b}\right)}{(x+b)}dx$$

These forms are now integrable by the table of integrals, the result being

$$\int \frac{1}{(x+a)(x+b)}dx = \frac{1}{b-a}\log(x+a) + \frac{1}{a-b}\log(x+b) + C$$

where C is the constant of integration.

Integration by Parts

To integrate by parts, apply the formula

$$\int u\,dv = uv - \int v\,du$$

The method of integration by parts is most effective in dealing with the integration of **products,** involving logarithms, and trigonometric and inverse circular functions.

Generally, the most complicated quantity which can be integrated directly by one of the fundamental formulæ (see Table of Integrals, page 46) is equated, with the differential, to dv, and the remaining part is equated to u.

Example. To find

$$\int x \log(x)\,dx$$

Let $\qquad u = \log x \qquad$ and $\qquad dv = x\,dx$

then $\qquad du = \dfrac{dx}{x} \qquad v = \int x\,dx = \dfrac{x^2}{2}$

Substituting in the formula

$$\int u\,dv = uv - \int v\,du$$

we have

$$\int x \log(x)\,dx = \log(x) \cdot \frac{x^2}{2} - \int \frac{x^2}{2}\,\frac{dx}{x}$$

$$= \frac{x^2}{2} \log(x) - \frac{x^2}{4} + C$$

Integration by Substitution

I. Differentials containing fractional powers of x may be integrated by the substitution

$$x = z^n$$

where n is the least common denominator of the fractional exponents of x.

II. Expressions involving only fractional powers of $(a + bx)$ may be rationalized by the substitution

$$(a + bx) = z^n$$

where n is the least common denominator of the fractional exponents of $(a + bx)$.

III. To integrate expressions containing

$$\sqrt{x^2 + ax + b},$$

use the substitution

$$\sqrt{x^2 + ax + b} = z - x$$

IV. Expressions containing $\sqrt{-x^2 + ax + b}$ may be rationalized by the substitution

$$\sqrt{-x^2 + ax + b} = (x - \theta)\, z$$

where $(x - \theta)$ is a factor of $(-x^2 + ax + b)$.

V. A differential containing $\sin x$ and $\cos x$ can be transformed by means of the substitution

$$\tan \frac{x}{2} = z$$

from which

$$\sin x = \frac{2\,z}{1 + z^2} \qquad \cos x = \frac{1 - z^2}{1 + z^2} \qquad dx = \frac{2\,dz}{1 + z^2}$$

VI. A very useful substitution is

$$x = \frac{1}{z}$$

VII. Differentials involving $\sqrt{a^2 - x^2}$ may be rationalized by the substitution

$$x = a \sin \theta$$

VIII. Differentials involving $\sqrt{a^2 + x^2}$ may be rationalized by the substitution

$$x = a \tan \theta$$

IX. Differentials involving $\sqrt{x^2 - a^2}$ may be rationalized by the substitution

$$x = a \sec \theta$$

Reduction Formulæ

The purpose of the following reduction formulæ is to simplify an integral of the form

$$\int x^m (a + bx^n)^p \, dx$$

$$\int x^m (a + bx^n)^p \, dx = \frac{x^{m-n+1} (a + bx^n)^{p+1}}{(np + m + 1) b} - \frac{(m - n + 1) a}{(np + m + 1) b} \int x^{m-n} (a + bx^n)^p \, dx$$

This formula enables us to lower the exponent of x by n, without affecting the exponent of $(a + bx^n)$.

Method fails when $(np + m + 1) = 0$.

II. $$\int x^m (a + bx^n)^p \, dx = \frac{x^{m+1} (a + bx^n)^p}{(np + m + 1)} + \frac{npa}{(np + m + 1)} \int x^m (a + bx^n)^{p-1} \, dx$$

By this formula, the exponent of $(a + bx^n)$ is lowered by 1, without affecting the exponent of x.

Method fails when $(np + m + 1) = 0$.

III. $$\int x^m (a + bx^n)^p \, dx = \frac{x^{m+1} (a + bx^n)^{p+1}}{(m + 1)a} - \frac{(np + m + 1 + n) b}{(m + 1) a} \int x^{m+n} (a + bx^n)^p \, dx$$

By this formula, the exponent of x is increased by n, without affecting the exponent of $(a + bx^n)$.

Method fails when $m = -1$.

IV. $$\int x^m (a + bx^n)^p \, dx = -\frac{x^{m+1}(a + bx^n)^{p+1}}{n(p+1)a} + \frac{(np + n + m + 1)}{n(p+1)a} \int x^m (a + bx^n)^{p+1} \, dx$$

This formula enables us to increase the exponent of $(a + bx^n)$ by 1, without affecting the exponent of x. Method fails when $p = -1$.

TABLE OF INTEGRALS

Fundamental Forms

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

$$\int \frac{dx}{x} = \log x$$

$$\int e^x \, dx = e^x$$

$$\int a^x \, dx = \frac{a^x}{\log_e a}$$

$$\int \frac{dx}{1 + x^2} = \tan^{-1} x$$

$$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x$$

$$\int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x$$

$$\int \sin x \, dx = -\cos x$$

$$\int \cos x \, dx = \sin x$$

$$\int \tan x \, dx = \log(\sec x)$$

$$\int \cot x \, dx = \log(\sin x)$$

$$\int \sec x\,dx = \log\left[\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right]$$

$$\int \operatorname{cosec} x\,dx = \log\left(\tan\frac{x}{2}\right)$$

$$\int \tan x \sec x\,dx = \sec x$$

$$\int \cot x \operatorname{cosec} x\,dx = -\operatorname{cosec} x$$

$$\int \sec^2 x\,dx = \tan x$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x$$

Expressions involving $(a + bx)$:

$$\int \frac{dx}{(a+bx)} = \frac{1}{b}\log(a+bx)$$

$$\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}$$

$$\int \frac{x\,dx}{(a+bx)} = \frac{1}{b^2}[a+bx-a\log(a+bx)]$$

$$\int \frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\left[\log(a+bx)+\frac{a}{a+bx}\right]$$

$$\int \frac{dx}{x(a+bx)} = -\frac{1}{a}\log\frac{a+bx}{x}$$

$$\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\log\frac{a+bx}{x}$$

$$\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax}+\frac{b}{a^2}\log\frac{a+bx}{x}$$

Expressions involving $(a + bx^2)$ or $(a^2 \pm x^2)$:

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}$$

$$\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}} \tan^{-1}\left(x\sqrt{\frac{b}{a}}\right)$$

or $$\int \frac{dx}{a+bx^2} = \frac{1}{2\sqrt{-ab}} \log \frac{\sqrt{a}+x\sqrt{-b}}{\sqrt{a}-x\sqrt{-b}} \quad \text{if } a>0,\ b<0$$

$$\int \frac{dx}{(a+bx^2)^2} = \frac{x}{2a(a+bx^2)} + \frac{1}{2a}\int \frac{dx}{a+bx^2}$$

$$\int \frac{x\,dx}{a+bx^2} = \frac{1}{2b} \log\left(x^2+\frac{a}{b}\right)$$

$$\int \frac{dx}{x(a+bx^2)} = \frac{1}{2a} \log \frac{x^2}{a+bx^2}$$

Expressions involving $\sqrt{a+bx}$:

$$\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3}$$

$$\int x\sqrt{a+bx}\,dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}$$

$$\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}$$

$$\int \frac{\sqrt{a+bx}}{x}\,dx = 2\sqrt{a+bx} + a\int \frac{dx}{x\sqrt{a+bx}}$$

$$\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}$$

$$\int \frac{x\,dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2}\sqrt{a+bx}$$

$$\int \frac{x^2\,dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx}$$

$$\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}} \log\left[\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right]$$

or

$$\int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}} \tan^{-1}\sqrt{\frac{a+bx}{-a}}$$

$$\int \frac{dx}{x^2\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{a+bx}}$$

Expressions involving $\sqrt{a^2-x^2}$ or $\sqrt{a^2+x^2}$:

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right]$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a}\left[\log\frac{a+\sqrt{a^2\pm x^2}}{x}\right]$$

$$\int \frac{\sqrt{a^2\pm x^2}}{x}\,dx = \sqrt{a^2\pm x^2} - a\log\left[\frac{a+\sqrt{a^2\pm x^2}}{x}\right]$$

$$\int \frac{x\,dx}{\sqrt{a^2\pm x^2}} = \pm\sqrt{a^2\pm x^2}$$

$$\int x\sqrt{a^2-x^2}\,dx = -\tfrac{1}{3}\sqrt{(a^2-x^2)^3}$$

$$\int \sqrt{(a^2-x^2)^3}\,dx = \frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2} + \frac{3}{8}a^4\sin^{-1}\frac{x}{a}$$

$$\int x^2\sqrt{a^2-x^2}\,dx = -\frac{x}{4}\sqrt{(a^2-x^2)^3} + \frac{a^2}{8}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right]$$

$$\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}$$

$$\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x} - \sin^{-1}\frac{x}{a}$$

Expressions involving $\sqrt{x^2+a^2}$ or $\sqrt{x^2-a^2}$:

$$\int \sqrt{x^2\pm a^2}\,dx = \tfrac{1}{2}\left[x\sqrt{x^2\pm a^2} \pm a^2\log\left(x+\sqrt{x^2\pm a^2}\right)\right]$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\left[x + \sqrt{x^2 \pm a^2}\right]$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\cos^{-1}\frac{a}{x}$$

$$\int \frac{\sqrt{x^2 - a^2}}{x}\,dx = \sqrt{x^2 - a^2} - a\cos^{-1}\frac{a}{x}$$

$$\int \frac{x\,dx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2}$$

$$\int x\sqrt{x^2 \pm a^2}\,dx = \tfrac{1}{3}\sqrt{(x^2 \pm a^2)^3}$$

$$\int \sqrt{(x^2 \pm a^2)^3}\,dx = \frac{x}{8}(2x^2 \pm 5a^2)\sqrt{x^2 \pm a^2} + \frac{3a^4}{8}\log\left(x + \sqrt{x^2 \pm a^2}\right)$$

$$\int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}$$

$$\int x^2\sqrt{x^2 \pm a^2}\,dx = \frac{x}{8}(2x^2 \pm a^2)\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\log\left(x + \sqrt{x^2 \pm a^2}\right)$$

$$\int \frac{x^2\,dx}{\sqrt{x^2 \pm a^2}} = \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2}\log\left(x + \sqrt{x^2 \pm a^2}\right)$$

$$\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp\frac{\sqrt{x^2 \pm a^2}}{a^2 x}$$

$$\int \frac{\sqrt{x^2 \pm a^2}\,dx}{x^2} = -\frac{\sqrt{x^2 \pm a^2}}{x} + \log\left(x + \sqrt{x^2 \pm a^2}\right)$$

Expressions involving $\sqrt{\pm ax^2 + bx + c}$:

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\log\left(2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right)$$

$$\int \sqrt{ax^2 + bx + c}\, dx = \frac{2ax + b}{4a}\sqrt{ax^2 + bx + c} - \frac{b^2 - 4ac}{8a}\int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

$$\int \frac{dx}{\sqrt{-ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \sin^{-1}\left(\frac{2ax - b}{\sqrt{b^2 + 4ac}}\right)$$

$$\int \sqrt{-ax^2 + bx + c}\, dx = \frac{2ax - b}{4a}\sqrt{-ax^2 + bx + c} + \frac{b^2 + 4ac}{8a}\int \frac{dx}{\sqrt{-ax^2 + bx + c}}$$

Formulæ involving $\sqrt{2ax - x^2}$:

$$\int \sqrt{2ax - x^2}\, dx = \frac{x - a}{2}\sqrt{2ax - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x - a}{a}$$

$$\int x\sqrt{2ax - x^2}\, dx = -\frac{3a^2 + ax - 2x^2}{6}\sqrt{2ax - x^2} + \frac{a^3}{2}\operatorname{vers}^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{2ax - x^2}} = \operatorname{vers}^{-1}\frac{x}{a}$$

$$\int \frac{x\, dx}{\sqrt{2ax - x^2}} = -\sqrt{2ax - x^2} + a \operatorname{vers}^{-1}\frac{x}{a}$$

$$\int \frac{dx}{x\sqrt{2ax - x^2}} = -\frac{\sqrt{2ax - x^2}}{ax}$$

$$\int \frac{\sqrt{2ax - x^2}}{x}\, dx = \sqrt{2ax - x^2} + a \operatorname{vers}^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{(2ax - x^2)^3}} = \frac{x - a}{a^2\sqrt{2ax - x^2}}$$

$$\int \sqrt{\frac{a + x}{b + x}}\, dx = \sqrt{(a + x)(b + x)} + (a - b)\log\left[\sqrt{a + x} + \sqrt{b + x}\right]$$

$$\int \sqrt{\frac{a-x}{b+x}}\,dx = \sqrt{(a-x)(b+x)} + (a+b)\sin^{-1}\sqrt{\frac{b+x}{a+b}}$$

Expressions involving trigonometric forms:

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin(2x)$$

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin(2x)$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

$$\int \sin x \cos x\,dx = \tfrac{1}{2}\sin^2 x$$

$$\int \sin^2 x \cos^2 x\,dx = -\tfrac{1}{8}\left[\tfrac{1}{4}\sin(4x) - x\right]$$

$$\int \sin x \cos^m x\,dx = -\frac{\cos^{m+1} x}{m+1}$$

$$\int \sin^m x \cos x\,dx = \frac{\sin^{m+1} x}{m+1}$$

$$\int \cos^m x \sin^n x\,dx = \frac{\cos^{m-1} x \sin^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \cos^{m-2} x \sin^n x\,dx$$

$$\int \cos^m x \sin^n x\,dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n}\int \cos^m x \sin^{n-2} x\,dx$$

$$\int \frac{\sin^m x}{\cos^n x}\,dx = \frac{\sin^{m+1} x}{(n-1)\cos^{n-1} x} + \frac{n-m-2}{n-1}\int \frac{\sin^m x}{\cos^{n-2} x}\,dx$$

$$\int \frac{\cos^n x}{\sin^m x}\,dx = -\frac{\cos^{n+1} x}{(m-1)\sin^{m-1} x} + \frac{m-n-2}{m-1}\int \frac{\cos^n x}{\sin^{m-2} x}\,dx$$

$$\int \frac{dx}{\sin^m x} = -\frac{\cos x}{(m-1)\sin^{m-1} x} + \frac{m-2}{m-1}\int \frac{dx}{\sin^{m-2} x}$$

$$\int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x}$$

$$\int \tan x\,dx = -\log\cos x$$

$$\int \tan^2 x\,dx = \tan x - x$$

$$\int \cot x\,dx = \log\sin x$$

$$\int \cot^2 x\,dx = -\cot x - x$$

$$\int \sec x\,dx = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \frac{1}{2}\log\frac{1+\sin x}{1-\sin x}$$

$$\int \sec^2 x\,dx = \tan x$$

$$\int \operatorname{cosec} x\,dx = \log\tan\left(\tfrac{1}{2}x\right)$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x$$

$$\int x\sin x\,dx = \sin x - x\cos x$$

$$\int x^2\sin x\,dx = 2x\sin x - (x^2-2)\cos x$$

$$\int x\cos x\,dx = \cos x + x\sin x$$

$$\int x^2\cos x\,dx = 2x\cos x + (x^2-2)\sin x$$

Transcendentals

$$\int \log x \, dx = x \log x - x$$

$$\int \frac{(\log x)^n}{x} dx = \frac{1}{n+1} (\log x)^{n+1}$$

$$\int \frac{dx}{x \log x} = \log \log x$$

$$\int \frac{dx}{x\,(\log x)^n} = -\frac{1}{(n-1)\,(\log x)^{n-1}}$$

$$\int x^m \log x \, dx = x^{m+1} \left[\frac{\log x}{m+1} - \frac{1}{(m+1)^2} \right]$$

$$\int x\, e^{ax} \, dx = \frac{e^{ax}}{a^2} (ax - 1)$$

$$\int x^m\, e^{ax} \, dx = \frac{x^m\, e^{ax}}{a} - \frac{m}{a} \int x^{m-1}\, e^{ax} \, dx$$

$$\int \frac{e^{ax}}{x^m} dx = -\frac{1}{m-1} \frac{e^{ax}}{x^{m-1}} + \frac{a}{m-1} \int \frac{e^{ax}}{x^{m-1}} dx$$

$$\int e^{ax} \sin (nx) \, dx = e^{ax} \left[\frac{a \sin (nx) - n \cos (nx)}{a^2 + n^2} \right]$$

$$\int e^{ax} \cos (nx) \, dx = e^{ax} \left[\frac{a \cos (nx) + n \sin (nx)}{a^2 + n^2} \right]$$

HYPERBOLIC FUNCTIONS

Hyperbolic Transformations

$$\sinh x = \frac{e^x - e^{-x}}{2} = -j \sin (jx)$$

where $$j = \sqrt{-1}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \cos (jx)$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = -j \tan (jx)$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = j \cot (jx)$$

$$e^x = \cosh x + \sinh x$$

$$e^{-x} = \cosh x - \sinh x$$

$$\sin x = -j \sinh (jx)$$

$$\cos x = \cosh (jx)$$

Hyperbolic Formulæ

$$\cosh^2 x - \sinh^2 x = 1$$

$$\operatorname{sech}^2 x + \tanh^2 x = 1$$

$$\coth^2 x - \operatorname{cosech}^2 x = 1$$

$$\sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh (x + y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\sinh (x - y) = \sinh x \cosh y - \cosh x \sinh y$$

$$\cosh (x - y) = \cosh x \cosh y - \sinh x \sinh y$$

$$\tanh (x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

$$\coth (x + y) = \frac{\coth x \coth y + 1}{\coth y + \coth x}$$

$$\tanh (x - y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$$

$$\coth (x - y) = \frac{\coth x \coth y - 1}{\coth y - \coth x}$$

$$\sinh (2x) = 2 \sinh x \cosh x$$

$$\cosh (2x) = \cosh^2 x + \sinh^2 x$$

$$\tanh (2x) = \frac{2 \tanh x}{1 + \tanh^2 x}$$

$$\coth(2x) = \frac{\coth^2 x + 1}{2\coth x}$$

$$\sinh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh x - 1}{2}}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh x + 1}{2}}$$

$$\tanh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh x - 1}{\cosh x + 1}}$$

$$\coth\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh x + 1}{\cosh x - 1}}$$

$$\sinh x + \sinh y = 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$$

$$\sinh x - \sinh y = 2\cosh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$$

$$\cosh x + \cosh y = 2\cosh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$$

$$\cosh x - \cosh y = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$$

$$\sinh(3x) = 3\sinh x + 4\sinh^3 x$$

$$\cosh(3x) = -3\cosh x + 4\cosh^3 x$$

Inverse Hyperbolic Functions

$$\sinh^{-1} x = \log\left(x + \sqrt{1 + x^2}\right)$$

$$\cosh^{-1} x = \log\left(x + \sqrt{x^2 - 1}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\log\left[\frac{1+x}{1-x}\right]$$

$$\coth^{-1} x = \frac{1}{2}\log\left[\frac{x+1}{x-1}\right]$$

$$\operatorname{sech}^{-1} x = \log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right)$$

$$\operatorname{cosech}^{-1} x = \log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right)$$

Differentials of Hyperbolic Functions

$$d(\sinh x) = \cosh x \, dx$$

$$d(\cosh x) = \sinh x \, dx$$

$$d(\tanh x) = \operatorname{sech}^2 x \, dx$$

$$d(\coth x) = - \operatorname{cosech}^2 x \, dx$$

$$d(\operatorname{sech} x) = - \operatorname{sech} x \tanh x \, dx$$

$$d(\operatorname{cosech} x) = - \operatorname{cosech} x \coth x \, dx$$

$$d(\sinh^{-1} x) = \frac{dx}{\sqrt{1 + x^2}}$$

$$d(\cosh^{-1} x) = \frac{dx}{\sqrt{x^2 - 1}}$$

$$d(\tanh^{-1} x) = \frac{dx}{1 - x^2}$$

$$d(\coth^{-1} x) = \frac{dx}{1 - x^2}$$

$$d(\operatorname{sech}^{-1} x) = - \frac{dx}{x\sqrt{1 - x^2}}$$

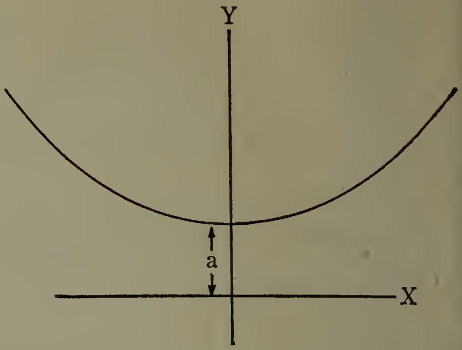
$$d(\operatorname{cosech}^{-1} x) = - \frac{dx}{x\sqrt{x^2 + 1}}$$

Use of Hyperbolic Functions

Illustrative Example. Deduce an expression for the length of a perfectly flexible chain suspended between two supports; assume that both points of support are the same height from the ground.

The chain assumes the form of a catenary (see page 27), the equation of which is

$$y = a \cosh \frac{x}{a}$$



The general equation for the length of the chain is

$$L = \text{length} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

where the value of $\frac{dy}{dx}$, obtained by differentiating the equation of the catenary, is

$$\frac{dy}{dx} = \frac{d\left(a \cosh \frac{x}{a}\right)}{dx} = a\left[\left(\sinh \frac{x}{a}\right)\left(\frac{1}{a}\right)\right] = \sinh \frac{x}{a}$$

Substituting the value of $\frac{dy}{dx}$ in the formula for the length, L, we have

$$L = \int \sqrt{1 + \sinh^2 \frac{x}{a}}\, dx = \int \sqrt{\cosh^2 \frac{x}{a}}\, dx$$

$$= \int \cosh \frac{x}{a}\, dx = a \sinh \frac{x}{a}$$

which is the required expression for the length of the chain.

DIFFERENTIAL EQUATIONS

A **differential equation** is a relation involving derivatives or differentials.

A **solution** of a differential equation is a relation

between the variables which satisfies the given equation.

ORDINARY DIFFERENTIAL EQUATIONS

Equations of the First Order and First Degree

I. An equation of the form

$$f_1(x)\,dx + f_2(y)\,dy = 0$$

can be integrated immediately.

Its solution is

$$\int f_1(x)\,dx + \int f_2(y)\,dy = C$$

An equation may sometimes be changed to the above form by separation of the variables.

II. **Homogeneous Equation.** An equation is **homogeneous** in respect to its variables when the sum of their exponents is the same for each term of the equation.

Homogeneous equations are reduced to the form of Method I, by substituting vx for y, and then separating the variables.

III. **Non-homogeneous Equation of First Degree in x and y.** This type occurs in the form:

$$(ax + by + c)\,dx = (a'x + b'y + c')\,dy$$

Substitute for x, $(x' + h)$, and for y, $(y' + k)$. The equation then becomes:

$$(ax' + by' + ah + bk + c)dx' = (a'x' + b'y' + a'h + b'k + c')dy'$$

Equating $\quad ah + bk + c = 0$

and $\quad a'h + b'k + c' = 0$

the original equation now takes the form:

$$(ax' + by')\,dx' = (a'x' + b'y')\,dy'$$

which is homogeneous and solvable by Method II.

In the solution thus obtained, substitute

$$x' = x - h \qquad \text{and} \qquad y' = y - k$$

where h and k are determined from the two equations:

$$ah + bk + c = 0$$
$$a'h + b'k + c' = 0$$

IV. **Linear Equation.** A linear differential equation (of first order and first degree) is of the general form:

$$\frac{dy}{dx} + Py = Q$$

where P and Q are functions of x alone or constants.

The solution of this equation is:

$$ye^{\int P\,dx} = \int e^{\int P\,dx} Q\,dx + C$$

V. **Equations Reducible to the Linear Equation.** This type occurs in the form:

$$\frac{dy}{dx} + Py = Qy^n$$

where P and Q are functions of x alone. The given equation may be written:

$$\frac{dv}{dx} + (1 - n)\,Pv = (1 - n)\,Q$$

where $v = y^{-n+1}$. This equation is linear in v, and solvable by Method IV. In the solution, resubstitute for v its value y^{-n+1}.

VI. **Exact Differential Equation.** An equation of the form

$$M\,dx + N\,dy = 0$$

is exact if the derivative of M with regard to y is equal to the derivative of N with regard to x. The solution then is:

$$\int M\,dx + \int\left[N - \frac{\partial}{\partial y}\int M\,dx\right]dy = C$$

where $\int M\,dx$ is the integral of M with respect to x (regarding y as constant), and the term

$$\left[N - \frac{\partial}{\partial y}\int M\,dx\right]$$

is found by subtracting from N the derivative in respect to y of $\int M\,dx$. The term $\left[N - \frac{\partial}{\partial y}\int M\,dx\right]$ is integrated with regard to y (considering x constant). The complete solution is then given by the formula above.

VII. **Integrating Factors.** If a differential equation of the form

$$M\,dx + N\,dy = 0$$

is multiplied through by a certain expression called an integrating factor, the equation will become exact. It is then solvable by Method VI.

(a) When an equation is homogeneous, $\dfrac{1}{Mx + Ny}$ is an integrating factor.

(b) When the condition exists that

$$\frac{\dfrac{dM}{dy} - \dfrac{dN}{dx}}{N} = F(x) \quad \text{[an expression containing only } x\text{]}$$

then $e^{\int F(x)\,dx}$ is an integrating factor.

(c) Similarly when

$$\frac{\dfrac{dN}{dx} - \dfrac{dM}{dy}}{M} = F(y)$$

then $e^{\int F(y)\,dy}$ is an integrating factor.

Equations of the First Order but Higher than the First Degree

In the following formulæ, $\dfrac{dy}{dx}$ will be denoted by p.

An equation of first order and of nth degree is of the general form

$$p^n + Ap^{n-1} + Bp^{n-2} + \cdots + Jp + K = 0$$

where the coefficients $A, B, \cdots J, K$ are functions of x and y.

I. **Clairaut's Equation.** When an equation is of the form

$$y = px + f(p)$$

the solution is obtained by substituting for p a constant c,

$$y = cx + f(c)$$

II. **Solution by Factoring.** The given equation may sometimes be resolved into rational factors of the first degree. Each factor is equated separately to zero, and its solution found by one of the preceding methods, using the same constant of integration in each case. The complete solution is then the product of the separate solutions.

III. **Equations Containing only x and p.** When an equation is of this type, solve for p, and substitute

its value $\frac{dy}{dx}$. The resulting equation can be integrated immediately.

IV. **Equations Containing only y and p.** Solve for p, and substitute its value $\frac{dy}{dx}$. This equation is immediately integrable.

V. **Equations Involving x, y, and p.** A solution can be obtained by one of the following methods:

(a) Solve for x in terms of y and p. Then differentiate in respect to y, remembering that $\frac{dx}{dy} = \frac{1}{p}$.

The solution of this equation, together with the given equation, constitutes the complete solution.

(b) Solve for y in terms of x and p. Differentiate with respect to x, and in place of $\frac{dy}{dx}$ substitute its value p. The complete solution consists of the solution of this equation, together with the original equation.

(c) Solve for p, and replace it with its value $\frac{dy}{dx}$. From this equation it may be possible to obtain a solution.

Linear Differential Equations with Constant Coefficients

A **linear differential equation** is of the first degree in the dependent variable and all of its derivatives.

The **particular integral** is the solution of the equation obtained without the introduction of constants of integration.

The **complementary function** is the solution obtained by temporarily equating to zero all those terms

of the equation that do not contain the dependent variable or derivatives thereof.

The **complete solution** is the sum of the particular integral and the complementary function.

A linear equation with **constant coefficients** is of the form:

$$\frac{d^ny}{dx^n} + P\frac{d^{n-1}y}{dx^{n-1}} + Q\frac{d^{n-2}y}{dx^{n-2}} + \cdots + Ry = X$$

where the coefficients $P, Q, \cdots R$ are constants; and X is a function of x. Replacing $\frac{d}{dx}$ by the symbol D, the equation becomes

$$(D^n + PD^{n-1} + QD^{n-2} + \cdots + R)\,y = X.$$

Case I. Method of Solution when X = 0. Write the given integral in its symbolic form, replacing $\frac{d}{dx}$ by D. Then solve this equation for D as if it were an ordinary algebraic quantity.

When the **roots** of the equation (i.e., the values of D) are **real,** the solution is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots$$

where c_1, c_2, etc., are the constants of integration, and m_1, m_2, etc., are the roots of the equation.

When two or more real roots of the equation are **equal,** the solution is

$$y = (c_1 + c_2x + c_3x^2 + \cdots)\,e^{mx} + \cdots$$

where m is the value of the repeated root, and c_1, c_2, c_3, etc., are the constants of integration (introduced in the manner shown in the above equation) and equal in number to the number of times the root m is repeated.

When the equation has **imaginary roots** (which always occur in pairs) the solution is

$$y = e^{m_1 x}[A \cos(a_1 x) + B \sin(a_1 x)] + e^{m_2 x}[C \cos(a_2 x) + D \sin(a_2 x)] + \cdots$$

where A and B, C and D, etc., are the constants of integration, and $(m_1 \pm a_1\sqrt{-1})$, $(m_2 \pm a_2\sqrt{-1})$, etc., are the complex imaginary roots of the equation.

When two or more pairs of complex imaginary roots are **equal,** the solution is

$$y = [(c_1 + c_2 x + \cdots) \cos(ax) + (c_3 + c_4 x + \cdots) \sin(ax)]\, e^{mx}$$

where $(m \pm a\sqrt{-1})$ is the repeated pair of complex imaginary roots.

Case II. Method of solution when X is not equal to zero. In this case, the **complete solution** is the **sum** of the complementary function and the particular integral.

The **complementary function** is found by temporarily equating $X = 0$, and obtaining the solution by the method of **Case I.**

The **particular integral** is obtained as follows.

The given equation is of the general form:

$$(D^n + PD^{n-1} + QD^{n-2} + \cdots + R)\, y = X$$

in which D is used in place of $\dfrac{d}{dx}$.

In symbolic notation, this equation may be expressed

$$f(D)\, y = X$$

The particular integral can then be written:

$$y = \frac{X}{f(D)} = \text{particular integral}$$

A. **Method of obtaining the particular integral when the term X is of the form e^{ax}.**

$$\text{particular integral} = \frac{X}{f(D)} = \frac{e^{ax}}{f(D)} = \frac{e^{ax}}{f(a)}$$

which is found by substituting the constant a in place of D.

This method for evaluating $\frac{e^{ax}}{f(D)}$ fails when the term $(D - a)$ is a factor of $f(D)$. The particular integral is then found by substituting the constant a for D in all terms of $f(D)$ except in the factor $(D - a)$. The solution is then completed by the general method given under case F (page 68).

B. **Solution for the particular integral when X has the form x^m.**

$$\text{particular integral} = \frac{X}{f(D)} = \frac{x^m}{f(D)} = [f(D)]^{-1} x^m$$

To evaluate this expression, expand $[f(D)]^{-1}$ into a series of ascending powers of D, by use of the binomial theorem. It is only necessary to carry out this expansion to the mth power of D, since operation on x^m by higher powers of D would produce zero (since the symbol D stands for $\frac{d}{dx}$, the operation by D on a quantity denotes its derivative with respect to x, the operation by D^2 denotes its second derivative, etc.). In obtaining the solution of the given particular integral, x^m is operated on separately by each term of the expansion of $[f(D)]^{-1}$.

C. **Method of obtaining the particular integral when X has the form sin (ax).**

$$\text{particular integral} = \frac{X}{f(D)} = \frac{\sin(ax)}{f(D)}$$

In order to evaluate this integral, substitute $-a^2$ for D^2 wherever D^2 occurs in $f(D)$. The particular integral will then be a fraction, whose numerator is $\sin(ax)$, and whose denominator is the value assumed by $f(D)$ when D^2 is replaced by $-a^2$.

This method fails if $f(D)$ becomes zero when $-a^2$ is substituted for D^2. The particular integral is then evaluated by writing the term e^{iax} (in which $i = \sqrt{-1}$) in place of $\sin(ax)$. The solution of this new integral is obtained by method A for the evaluation of the particular integral. In the result, e^{iax} is replaced by $[\cos(ax) + i\sin(ax)]$, producing a result containing both real and imaginary terms. The required particular integral is the coefficient of i (i.e., $\sqrt{-1}$) in this expression.

D. **Particular Integral when X = cos (ax).** The particular integral is obtained as in method C, with the exception that $\cos(ax)$ is used in place of $\sin(ax)$.

When this method fails, e^{iax} is written in place of $\cos(ax)$, and this new integral is evaluated by method A. In the solution of this integral, e^{iax} is replaced by $[\cos(ax) + i\sin(ax)]$. The required particular integral is the real part of this result.

E. **Particular integral when X is of the form $e^{ax}Q$.**

$$\text{particular integral} = \frac{X}{f(D)} = \frac{e^{ax}Q}{f(D)} = e^{ax}\frac{Q}{f(D+a)}$$

To evaluate the given integral, $(D + a)$ is substituted for D, wherever D occurs in $f(D)$; and the term e^{ax} is treated as a constant multiplier. The new integral $\dfrac{Q}{f(D+a)}$ is evaluated by one of the preceding methods, or by the general method F. The required particular

integral is then equal to the product of e^{ax} by the evaluation of $\dfrac{Q}{f(D+a)}$.

F. **General method for finding the particular integral.**

To evaluate $$\frac{1}{f(D)}X$$

The denominator of $\dfrac{1}{f(D)}$ may be resolved into factors of the first degree. The given integral then becomes:

$$\frac{1}{(D-a)}\,\frac{1}{(D-b)}\,\frac{1}{(D-c)}\,\frac{1}{(D-d)}\cdots\frac{1}{(D-m)}X$$

The term X is operated on successively by each of these fractional operators, beginning at the right. The operation on X by the first factor $\dfrac{1}{(D-m)}$ produces the expression $e^{mx}\displaystyle\int e^{-mx}X\,dx$. This result is operated on in a similar manner by each remaining factor (proceeding from right to left). The solution of the given particular integral is then:

$$e^{ax}\int e^{-ax}e^{bx}\int e^{-bx}e^{cx}\int e^{-cx}\ldots e^{mx}\int e^{-mx}X(dx)^m$$

Homogeneous Linear Equation

The homogeneous linear equation is of the form

$$x^n\frac{d^ny}{dx^n}+Px^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+Ry=X$$

in which the coefficients $P, \ldots R$ are constants, and X is a function of x.

On assuming the relation, $x = e^z$, this equation may be transformed by the substitutions:

$$x^n \frac{d^n y}{dx^n} = \theta(\theta - 1)(\theta - 2) \cdots \text{ to } n \text{ terms}$$

$$x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} = (\theta - 1)(\theta - 2)(\theta - 3) \cdots \text{ to } (n-1) \text{ terms,}$$

and so forth; where the symbol θ stands for $\frac{d}{dz}$.

The **complementary function** is then found as in the case of the linear equation with constant coefficients. (In obtaining this solution, the term θ is treated in exactly the same manner in which the term D was treated in the preceding cases.)

In order to obtain the **particular integral,** the term X (which involves only x) is changed to an expression involving z, by the substitution $x = e^z$. The particular integral is then found by one of the methods given under the case of the linear equation with constant coefficients.

The **complete solution** is the sum of the complementary function and the particular integral. In the result, z is replaced by its value $\log x$.

Exact Differential Equations

An exact differential equation is one which can be derived directly by differentiation of an equation of the next lower order.

If the given equation is of the form:

$$A \frac{d^n y}{dx^n} + B \frac{d^{n-1} y}{dx^{n-1}} + \cdots + Q \frac{d^3 y}{dx^3} + R \frac{d^2 y}{dx^2} + S \frac{dy}{dx} + Ty = X$$

where A, B, . . . Q, R, S, T, and X are functions of x, we then have as the condition for exactness that:

$$T - \frac{dS}{dx} + \frac{d^2R}{dx^2} - \frac{d^3Q}{dx^3} + \cdots = 0$$

The first integral of the given equation then is:

$$A\frac{d^{n-1}y}{dx^{n-1}} + \left(B - \frac{dA}{dx}\right)\frac{d^{n-2}y}{dx^{n-2}} + \left(C - \frac{dB}{dx} + \frac{d^2A}{dx^2}\right)\frac{d^{n-3}y}{dx^{n-3}} \cdots$$
$$= \int X\,dx + C$$

This formula may be reapplied successively as long as each resulting equation satisfies the condition for exactness.

Equations of the **Second Order** and the **First Degree**

General form is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X$$

where P, Q, and X are functions of x.

I. **When one solution of the equation is known (or can be found by inspection).**

Let y_1 equal the known integral. In the given equation, substitute vy_1 in place of y; and then, in the transformed equation, replace $\frac{dv}{dx}$ by p. This equation can be solved by one of the preceding methods.

II. **Change of the Independent Variable.**

The purpose of this change and of the removal of the first derivative (see III) is to transform a given equation into a new equation which may happen to be easily integrable.

The given equation is of the form:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X$$

By changing the independent variable, it may be transformed into the following equation:

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = X_1$$

where Q_1 becomes equal to 1, if

$$\frac{dz}{dx} = \sqrt{Q}$$

when also $$P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{Q}$$

and $$X_1 = \frac{X}{Q}$$

or where P_1 may be made equal to zero, if

$$z = \int e^{-\int P\,dx}\,dx$$

when also $$Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2}$$

and $$X_1 = \frac{X}{\left(\dfrac{dz}{dx}\right)^2}$$

III. Removal of the First Derivative.

To remove the first derivative from an equation of the form $$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X$$

make the substitution $y = ve^{-\frac{1}{2}\int P\,dx}$

The given equation then becomes

$$\frac{d^2v}{dx^2} + Q_1 v = X_1$$

where $$Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$$

and $$X_1 = Xe^{\frac{1}{2}\int P\,dx}$$

THEORETICAL MECHANICS

Center of Gravity

The **center of gravity** of a body is a point so situated that the force of gravity produces no tendency in the body to rotate about any axis passing through this point.

Center of Gravity of the Arc of a Plane Curve

$$\bar{x} = \frac{\int x\,ds}{\int ds} = \frac{\int x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

$$\bar{y} = \frac{\int y\,ds}{\int ds} = \frac{\int y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy}{\int \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy}$$

where $\bar{x}$ and $\bar{y}$ are the coördinates of the center of gravity.

Solve for y in terms of x from the equation of the given curve. Then differentiate in order to obtain $\frac{dy}{dx}$, and substitute its value in the formula for $\bar{x}$.

Similarly, find x in terms of y, obtain $\frac{dx}{dy}$, and substitute in the formula for $\bar{y}$.

Center of Gravity of Plane Areas

Rectangular Coördinates:

$$\bar{x} = \frac{\int\int x\,dA}{\int\int dA} = \frac{\int\int x\,dx\,dy}{\int\int dx\,dy}$$

$$\bar{y} = \frac{\int\int y\,dA}{\int\int dA} = \frac{\int\int y\,dx\,dy}{\int\int dx\,dy}$$

where $\bar{x}$ and $\bar{y}$ are the coördinates of the center of gravity.

In evaluating the expression for $\bar{x}$, we may integrate first either in respect to x or y, according to which method is more convenient.

If dy is integrated first, the limits of y are expressed in terms of x (from the given equation); and the limits of x are its initial and final values.

Similarly, if dx is first integrated, the limits of x are expressed in terms of y; and the limits of y are then its initial and final values.

Polar Coördinates:

$$\bar{x} = \frac{\int\int r^2 \cos\theta\, d\theta\, dr}{\int\int r\, d\theta\, dr}$$

$$\bar{y} = \frac{\int\int r^2 \sin\theta\, d\theta\, dr}{\int\int r\, d\theta\, dr}$$

Generally, it is more convenient to integrate first with respect to r. In this case, the limits of r are found in terms of θ from the equation of the given curve. The limits of θ are its initial and final values, expressed in radians.

Center of Gravity of Solids of Revolution. When a solid of uniform density is formed by the revolution of a plane curve about the X-axis, the center of gravity

is on the X-axis (because of symmetry). Its x-coördinate is

$$\bar{x} = \frac{\int\int xy\,dx\,dy}{\int\int y\,dx\,dy}$$

where the limits are found as in the case of plane areas.

When a solid is formed by the revolution of a plane figure about the Y-axis, the y-coördinate of its center of gravity is

$$\bar{y} = \frac{\int\int xy\,dx\,dy}{\int\int x\,dx\,dy}$$

Center of Gravity of Any Section Composed of Two or More Simple Plane Figures

In order to find the center of gravity of such figures as tee-bars, channels, rails, etc., divide them up into their component rectangles or triangles. Then, obtain the center of gravity and the area of each separate figure. Choose any convenient axis in the plane of the given section and find the turning moment of each figure about this axis. Each turning moment is the product of the area of the figure by the distance from its center of gravity to the chosen axis. The sum of all these separate turning moments gives the turning moment of the total figure. On dividing this total moment by the total area of the figure, we obtain the distance from the chosen axis to the center of gravity of the figure. Care must be used, if the chosen axis passes through the given figure, to take distances on one

side of this axis as positive, and on the other side as negative.

Generally, one coördinate of the center of gravity can be determined by the symmetry of the given section. When the figure is unsymmetrical, it may be necessary to take moments about two different axes in order to locate the center of gravity.

Moment of Inertia of Plane Areas

The **moment of inertia of a plane figure** about any given axis is equal to the integral of the product of each elementary area of the figure by the square of its distance from the axis.

Rectangular Moment of Inertia:

The **rectangular moment of inertia** of a plane figure is its moment of inertia about any axis in the plane of the figure. The rectangular moment of inertia of a plane area about the X-axis is

$$I_x = \int\int y^2\, dx\, dy$$

The rectangular moment of inertia of a plane area about the Y-axis is

$$I_y = \int\int x^2\, dx\, dy$$

In either case, the limits of the variable first integrated are expressed in terms of the other variable.

The **moment of inertia** of a plane figure **about the gravity axis** (I_g) is its rectangular moment of inertia about any axis in the plane of the figure, passing through its center of gravity.

The **moment of inertia** of a plane figure **about any axis parallel to the gravity axis** and in the plane of

the figure is equal to (I_g) plus the product of the area of the figure by the square of the distance between the two axes, thus:

$$I = I_g + Fd^2$$

Polar Moment of Inertia:

The **polar moment of inertia** (I_p) is the moment of inertia about any axis perpendicular to the plane of the given figure.

It is equal to the sum of the rectangular moments of inertia about two mutually perpendicular axes in the plane of the figure, passing through the foot of the polar axis.

In **rectangular coördinates,** the polar moment of inertia equals

$$I_p = I_x + I_y = \int\int (x^2 + y^2)\, dx\, dy$$

In **polar coördinates,** the formula for the polar moment of inertia is

$$I_p = \int\int R^3\, dR\, d\theta$$

It is generally more convenient to integrate first with respect to R, expressing its limits in terms of θ. The limits of θ are then its initial and final values.

Moment of Inertia of Solids

The moment of inertia of a solid (with center at origin) about the X-axis is

$$I = m\int\int\int (y^2 + z^2)\, dx\, dy\, dz$$

where m is the density, that is, the mass per unit volume.

Radius of Gyration

The **center of gyration** is that point in a revolving body at which, if the entire mass of the body were concentrated, the moment of inertia about the axis of rotation would be the same as that of the body.

The **radius of gyration,** k, is the distance from the axis of rotation to the center of gyration.

For plane sections, $k = \sqrt{\frac{I}{A}}$

For solids, $k = \sqrt{\frac{I}{M}} = \sqrt{\frac{I}{\left(\frac{W}{g}\right)}}$

in which k = radius of gyration,
I = the moment of inertia about the axis of rotation,
A = area of section,
M = mass of body,
W = weight of body.

Center of Percussion

The **center of percussion** or oscillation of a pendulum or other body vibrating or rotating about a fixed axis or center is that point at which, if the entire weight of the body were concentrated, the body would continue to vibrate in the same intervals of time.

The **radius of oscillation** is

$$h = \frac{I}{Md} = \frac{I}{\left(\frac{W}{g}\right)d}$$

in which I = the moment of inertia of body about axis of rotation,
d = distance from center of gravity of body to the axis of rotation,
h = distance from center of percussion or oscillation to the axis of rotation,
M = mass of body,
W = weight of body.

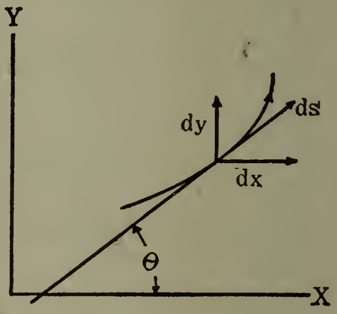
Motion of a Body

$$\text{velocity at any instant} = v = \frac{ds}{dt}$$

$$\text{acceleration at any instant} = a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

In rectangular coördinates,

$v_x = \dfrac{dx}{dt} = \dfrac{ds}{dt}\cos\theta$ = velocity in a direction parallel to the X-axis



$$v_y = \frac{dy}{dt} = \frac{ds}{dt}\sin\theta$$

$$v = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

For motion with **uniform velocity,**

$$v = \frac{s}{t}$$

For **uniformly accelerated motion,**

$$s = \tfrac{1}{2}(u + v)\,t$$
$$s = ut + \tfrac{1}{2}\,at^2$$
$$2\,as = v^2 - u^2$$

u = initial velocity,
v = final velocity,
a = constant acceleration,
s = space passed over,
t = time of motion.

If the **body starts from rest,** the initial velocity u equals 0, and these equations become:

$$s = \tfrac{1}{2}\,vt$$
$$s = \tfrac{1}{2}\,at^2$$
$$2\,as = v^2$$

Rotation of a Rigid Body

$$\text{velocity at any instant} = \omega = \frac{d\theta}{dt}$$

$$\text{acceleration at any instant} = \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

For motion with **uniform velocity,**

$$\omega = \frac{\theta}{t}$$

For **uniformly accelerated motion,**

$$\theta = \tfrac{1}{2}\,(\omega_0 + \omega)\,t$$
$$\theta = \omega_0 t + \tfrac{1}{2}\,\alpha t^2$$
$$2\,\alpha\theta = \omega^2 - \omega_0^2$$

θ = angular space through which the body rotates,
ω_0 = initial angular velocity,
ω = final angular velocity,
α = angular acceleration,
t = time.

For a **body initially at rest,** the velocity ω_0 is 0, and these equations become

$$\theta = \tfrac{1}{2}\,\omega t$$
$$\theta = \tfrac{1}{2}\,\alpha t^2$$
$$2\,\alpha\theta = \omega^2$$

Falling Bodies

Equations of motion of a **body falling from rest** under the action of gravity:

$$v = gt$$
$$s = \tfrac{1}{2}\,gt^2$$
$$2\,gs = v^2$$

v = velocity after time t,
s = height through which body falls,
g = (approx.) 32.16 feet/sec.^{2} = 981 cm/sec.^{2}
= acceleration of gravity.

The **value of g** for **any latitude** and **any altitude** is

$$g = 32.0894\,(1 + 0.0052375 \sin^2\theta) \times (1 - 0.0000000957\,E)$$

in which

θ = latitude of place in degrees,
E = elevation above sea-level in feet.

Projectiles

Equations of a body **projected vertically upward** with an initial velocity u (resistance of air not considered):

(1) Velocity at any time $= u - gt.$

(2) Velocity at any height $= \sqrt{u^2 - 2\,gh}.$

(3) Height at any time $= ut - \tfrac{1}{2}\,gt^2.$

(4) Greatest height $= \dfrac{u^2}{2g}$.

(5) Time of flight $= \dfrac{2u}{g}$.

Equations of a body **projected with an initial velocity** u **at an angle** $\theta°$ **to the horizontal** (resistance of air not considered):

The curve described by the projectile is the parabola whose equation is

$$y = x \tan\theta - \frac{gx^2}{2u^2 \cos^2\theta}$$

where θ is positive when the body is projected above the horizontal and negative when the body is projected below the horizontal.

Horizontal-component of acceleration $= \dfrac{d^2x}{dt^2} = 0$

Vertical-component of acceleration $= \dfrac{d^2y}{dt^2} = -g$

(1) Velocity at any time $= \sqrt{u^2 - 2utg \sin\theta + g^2t^2}$.

(2) Velocity at any height $= \sqrt{u^2 - 2gh}$.

(3) Height at any time $= ut \sin\theta - \frac{1}{2}gt^2$.

(4) Time of flight $= \dfrac{2u \sin\theta}{g}$.

(5) Range $= \dfrac{u^2 \sin(2\theta)}{g}$.

If the friction of the air is taken into account, the curve described by the projectile is given by the empirical relation:

$$y = x \tan\theta - \frac{gx^2}{2\cos^2\theta}\left(\frac{1}{u^2} + \frac{kx}{u}\right)$$

$$k = 0.0000000458 \frac{d^2}{w}$$

where

d = diameter of projectile in inches,
w = weight of projectile in pounds.

Angular Measure

A **radian** is the angle subtended at the center of any circle by an arc equal in length to its radius.

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees} = 57.296+ \quad \text{degrees}$$

$$1 \text{ degree} = \frac{\pi}{180} \text{ radians} = 0.0175+ \quad \text{radians}$$

The relation between the central angle of a circle and its subtended arc is given by the formula:

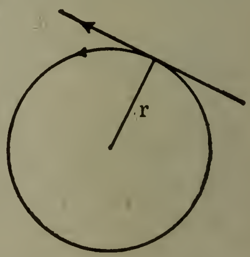
$$l = r\theta$$

l = length of arc,
r = radius of circle,
θ = central angle in radians.

Circular Motion

A body moving with **uniform velocity** in a circular path experiences a constant acceleration toward the center of the circle. This acceleration is expended in changing the direction of motion of the body.

The equations of motion of the revolving body are



$$a = \frac{v^2}{r}$$

$$vT = 2\pi r$$

$$a = \frac{4\pi^2 r}{T^2}$$

v = constant velocity of particle in feet per second,
a = constant acceleration toward center in feet per sec.^{2},

r = radius of circular path in feet,
T = time of 1 revolution in seconds,
$\pi^2 = 9.8696 +$.

If the body moves with a **variable velocity,** then:

$$\text{tangential acceleration} = \frac{dv}{dt}$$

$$\text{normal acceleration} = \frac{v^2}{r}$$

Centrifugal Force

The centrifugal force of a revolving body, in pounds, is

$$F = \frac{Wv^2}{gr} = \frac{4\pi^2 Wr}{gt^2}$$

or in terms of the number of revolutions, N_1, per minute

$$F = 0.00034\, WrN_1^2$$

W = weight of revolving body in pounds,
v = velocity of body in feet per second,
t = time of 1 revolution in seconds,
r = distance from axis of rotation to the center of gravity of the body, in feet,
g = acceleration of gravity (32.16).

Flywheel

The **energy** of rotation of a flywheel is

$$K.E. = \frac{I\omega^2}{2} = 2\pi^2 IN^2$$

I = polar moment of inertia about the axis of rotation,
ω = angular velocity in radians per second,
N = number of revolutions per second.

The **energy** stored in a rim flywheel by a variation in speed is

$$E = \frac{W}{2\,g}(S^2_{max} - S^2_{min}) \text{ foot-pounds,}$$

W = weight of flywheel in pounds,
S_{max} = maximum rim speed in feet per second,
S_{min} = minimum rim speed in feet per second
g = acceleration of gravity (32.16).

The rim speed in feet per second is $S = 2\,\pi RN$, where N is the speed in revolutions per second, and R is the radius of the wheel in feet, measured from the center of gravity of the rim section.*

Hence, the **energy stored** is

$$E = \frac{W}{g}\,\frac{4\,\pi^2R^2\,(N^2_{max} - N^2_{min})}{2} \text{ foot-pounds}$$

and the **weight** of the flywheel is

$$W = \frac{Eg}{2\,\pi^2R^2\,(N^2_{max} - N^2_{min})}$$

Substitute for E the required stored energy in foot-pounds. Assume some convenient value for R, in feet; then solve for the weight W in pounds. If the rim speed is too high (average about 35 feet per second for cast iron or 150 feet per second for steel), the value of R must be reduced. The ratio of the speed variation, $N_{max} - N_{min}$, to the average speed may be taken as follows for different types of machines:

Hammers	0.20
Punches	0.05
Ordinary machinery	0.03
Textile and paper machinery	0.02
Electric generators	0.005

* This value of R is approximately correct. The exact value of R is the radius of gyration of the flywheel.

Simple Pendulum

The **time of oscillation** in seconds from one extreme position to the other is

$$t = \pi\sqrt{\frac{l}{g}}$$

l = length of pendulum in feet,
g = acceleration of gravity (32.16 approx.).

The **period** of the pendulum is

$$P = 2t = 2\pi\sqrt{\frac{l}{g}}$$

The **seconds-pendulum** makes one oscillation per second from one extreme position to the other; its length in feet is

$$l = \frac{g}{\pi^2}$$

Work and Energy

For a **uniform** force,

$$F = ma \doteq \frac{W}{g}a$$

$$Ft = mv = \frac{W}{g}v$$

$$Fs = \tfrac{1}{2}mv^2 = \frac{Wv^2}{2g}$$

F = constant applied force in pounds,
a = constant acceleration in feet/sec.^{2},
m = mass of body,
W = weight of body in pounds,
v = velocity acquired after t seconds,
mv = momentum,
s = space passed over in feet,
g = acceleration of gravity (32.16 feet/sec.^{2}).

The **impulse** I of the constant force F during the time t equals the change of momentum,

$$I = Ft = mv - mu$$

where u is the initial velocity and v the final velocity.

If the force is variable, then **impulse** equals

$$I = \int_0^t F\,dt$$

The **work** done by a uniform force is

$$W = Fs = \tfrac{1}{2}\,mv^2$$

The **work** done by a variable force equals

$$W = \int_0^s F\,ds$$

The **kinetic energy** of a body of mass m, moving with a velocity v, equals $\frac{1}{2}\,mv^2$.

Direct Central Impact

For the impact of two bodies of the same material, weighing respectively W and W_1 pounds, the velocities after impact are

$$v = \frac{Wu + W_1u_1 - eW_1\,(u - u_1)}{W + W_1}$$

$$v_1 = \frac{Wu + W_1u_1 + eW\,(u - u_1)}{W + W_1}$$

u = original velocity of W in feet/second,
v = velocity of W after impact,
u_1 = original velocity of W_1,
v_1 = velocity of W_1 after impact,
e = coefficient of restitution.

Values of e, the coefficient of restitution, for different materials are as follows:

glass on glass..................	$e = 0.94$
ivory on ivory.................	$e = 0.81$
cast iron on cast iron..........	$e = 0.66$
lead on lead....................	$e = 0.2$

The sum of the momenta of two bodies after impact equals the sum of their momenta before impact,

$$\frac{Wv}{g} + \frac{W_1v_1}{g} = \frac{Wu}{g} + \frac{W_1u_1}{g}$$

Two **inelastic** bodies after impact move with a common velocity

$$v = \frac{W_1v_1 + W_2v_2}{W_1 + W_2}$$

in which

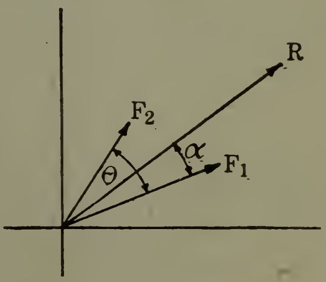
W_1 = weight of first body,
W_2 = weight of second body,
v_1 = original velocity of first body,
v_2 = original velocity of second body.

Composition and Resolution of Forces

The **resultant** of the forces F_1 and F_2 acting at a point is

$$R = \sqrt{F_1^2 + 2\,F_1F_2\cos\theta + F_2^2}$$

in which θ is the angle in degrees between the two forces.



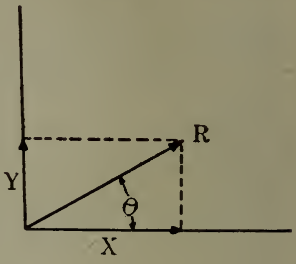
The **direction** of R is determined by the relation

$$\tan\alpha = \frac{F_2\sin\theta}{F_1 + F_2\cos\theta}$$

in which α is the angle in degrees between F_1 and R.

The **rectangular components** of a force R acting in a given direction are

$$X = R\cos\theta$$
$$Y = R\sin\theta$$

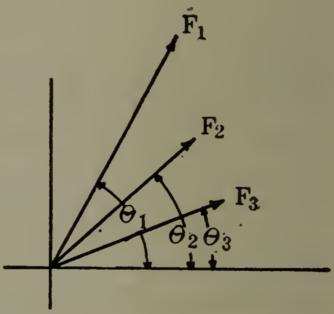


in which X is the horizontal component of R, Y is the normal component of R, and θ is the angle in degrees between R and X.

The **resultant** of several forces acting in different directions at a point is

$$R = \sqrt{X^2 + Y^2}$$

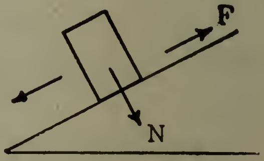
in which



$$X = F_1\cos\theta_1 + F_2\cos\theta_2 + F_3\cos\theta_3 + \cdots,$$
$$Y = F_1\sin\theta_1 + F_2\sin\theta_2 + F_3\sin\theta_3 + \cdots,$$

where F_1, F_2, F_3, etc., are the given forces, and θ_1, θ_2, θ_3, etc., are the angles in degrees between the given forces and the horizontal axis.

Friction



F = friction in pounds,
N = normal force in pounds,
f = coefficient of friction.

$$F = fN$$

$$\text{Angle of friction} = \phi = \tan^{-1}\frac{F}{N} = \tan^{-1}f$$

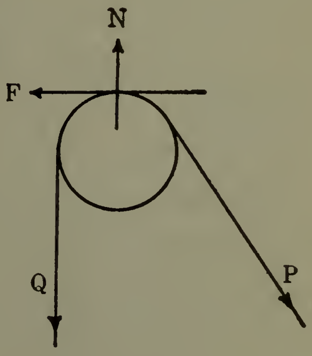
Average values for f, the coefficient of friction, for motion are as follows:

Character of contact	f
Wood on wood........................	0.25–0.50
Metal on wood........................	0.50–0.60
Metal on metal, dry.....................	0.15–0.24
Metal on metal, lubricated..............	0.075
Leather on metal, dry....................	0.56
Leather on metal, lubricated............	0.15

Belt Friction

P and Q are the forces at the ends of the belt, P being the greater force.

F = resultant force of friction,
N = normal reaction of pulley,
θ = angle in radians subtended by the arc of contact,
f = coefficient of friction.



$$\log_e \frac{P}{Q} = f\theta$$

or in common logarithms

$$\log_{10} \frac{P}{Q} = 0.434 f\theta$$

The value of f varies from 0.15 to 0.6 depending on the condition of belt and pulley, but, in general, it is approximately correct to assume $f = 0.3$.

Inclined Plane

Equations of motion of a body sliding down an incline under the action of its own weight.

For a frictionless plane:

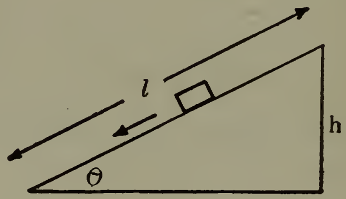
(1) acceleration along plane $= a = \dfrac{d^2s}{dt^2} = g \sin\theta,$

(2) velocity after t seconds $= tg \sin\theta,$

(3) velocity at bottom of plane $= \sqrt{2\,gh},$

(4) distance traveled in t seconds $= \dfrac{t^2 g \sin\theta}{2},$

(5) time of sliding down plane $= l\sqrt{\dfrac{2}{gh}}.$



For an inclined plane with friction:

(1) acceleration along plane $= a = \dfrac{d^2s}{dt^2}$

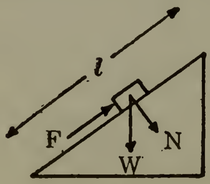
$$= g\,[\sin\theta - f\cos\theta],$$

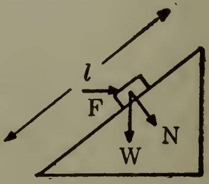
in which

f = coefficient of friction.

Conditions for the equilibrium of a body resting on an incline:

W = weight of body,
F = applied force,
N = normal pressure on plane,
θ = inclination of plane in degrees,
f = coefficient of friction.





For a frictionless plane:

(1) When the balancing force is applied parallel to the inclined plane,

$$F = W \sin \theta$$
$$N = W \cos \theta$$

(2) When the applied force acts horizontally,

$$F = W \tan \theta,$$
$$N = W \sec \theta.$$

For an inclined plane with friction:

(1) When the balancing force acts parallel to the incline,

$$F = \frac{W \sin (\theta \pm \theta')}{\cos (\theta')}$$

in which

$$\theta' = \tan^{-1} f$$

(2) When the applied force acts horizontally,

$$F = W \tan (\theta \pm \theta')$$

MECHANICS OF MATERIALS

Stress is distributed force; its intensity per unit area is generally expressed in pounds per square inch.

The **elastic limit** of a material is the maximum stress in pounds per square inch that will be followed by a complete recovery of form, after the removal of the stress.

Permanent set is the change in form of a member when stressed beyond its elastic limit.

The **ultimate strength** of a material is the least stress in pounds per square inch that will produce rupture.

Modulus of elasticity is the number obtained by dividing the actual stress in pounds per square inch by the corresponding elongation per inch.

The **factor of safety** is the factor obtained by dividing the ultimate strength by the actual stress in pounds per square inch.

Tension and Compression

For direct stress, uniformly distributed,

$$p = \frac{P}{F}$$

p = stress in pounds per square inch,
P = total load in pounds,
F = cross-sectional area in square inches.

$$E = \frac{p}{\epsilon} \qquad \epsilon = \frac{\lambda}{l}$$

$$E = \frac{\frac{P}{F}}{\frac{\lambda}{l}} = \frac{Pl}{F\lambda}$$

E = modulus of elasticity in tension or compression,
l = length of member in inches,
ϵ = elongation per inch length,
λ = total elongation in inches.

STRENGTH OF MATERIALS

Material	Density	Elastic limit	Ultimate strength			Modulus of elasticity		Factor of safety		
			Tension	Comp.	Shear	Tens. and comp.	Shear	Steady load	Var. load	Shocks
Brick	2	......	......	3,000	1,000	2,000,000	....	15	25	40
Stone	2.6	......	......	6,000	1,500	6,000,000	....	15	25	40
Timber	0.6	3,000	10,000	8,000	......	1,500,000	....	8	10	15
Timber along grain	0.6	......	......	......	500	....	....	....	......	......
Timber across grain	0.6	......	......	......	3,000	....	400,000	....	......	......
Cast iron	7.2	6,000	20,000	90,000	18,000	15,000,000	6,000,000	6	10	20
Wrought iron	7.7	25,000	50,000	50,000	40,000	25,000,000	10,000,000	4	6	10
Structural steel	7.8	35,000	60,000	60,000	50,000	30,000,000	12,000,000	4	6	10
Strong steel	7.8	50,000	100,000	120,000	80,000	30,000,000	12,000,000	5	8	15

Note.—The elastic limit of **6,000** for cast iron holds only for tension; for compression, the elastic limit is **20,000.**

Angular Distortion and Shear

Shearing stress, uniformly distributed equals

$$p_s = \frac{P}{F}$$

P = load,
F = area.

For **torsion:**

$$E_s = \frac{p_s}{\delta}$$

E_s = modulus of elasticity in shear,
δ = angle of distortion in radians.

Note. The modulus of elasticity in shear is $\frac{2}{5}$ as great as in compression or tension.

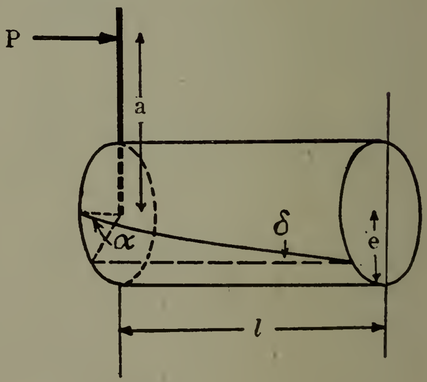
Torsion of Circular Shafts

$$\delta = \frac{e\alpha}{l} \qquad p_s = \frac{e\alpha E_s}{l}$$

$$Pa = \frac{p_s I_p}{e} = \frac{\alpha I_p E_s}{l}$$

$$I_p = \frac{\pi d^4}{32}$$

$$Pa = \frac{\pi p_s d^3}{16}$$



$$\text{Horsepower} = \frac{2\pi PaN}{33{,}000 \times 12}$$

δ = helix angle of distortion in radians,
α = radial angle of distortion in radians,
l = length of shaft in inches,
e = radius of shaft in inches,

p_s = greatest shearing stress in pounds per square inch existing in shaft,

E_s = modulus of elasticity in shear,

I_p = polar moment of inertia of circular section (see table of standard sections),

P = force in pounds producing torsion, that is, the turning force,

a = lever arm of force P in inches,

d = diameter of shaft in inches,

N = revolutions per minute.

In deriving the above formulæ, the torsion is treated as due to a couple of the same turning moment, Pa, as the single force P with lever arm a. This eliminates the consideration of any stresses other than shearing stresses, and, in applying these formulæ to the case of a single driving force, bending stresses and bearing friction are neglected.

Flexure of Beams

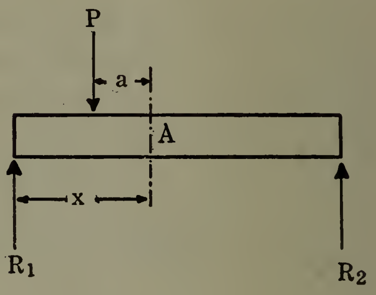
When a beam is strained by a vertical load, the greatest strain will be in the extreme upper and lower fibers of the beam. The intensity of the strain that can be borne by the extreme fibers is the limit of the strength of the beam. The upper fibers are compressed and the lower fibers are stretched when a beam is loaded between supports; the converse holds when it is loaded beyond supports. Somewhere along or near the center of the beam the fibers are neither extended nor compressed; the plane of these fibers is called the **neutral surface.** The line of intersection of the neutral surface with any cross-section of the beam is the **neutral axis** of the section.

If the stresses remain within the elastic limits of the material in both tension and compression, and provided the modulus of elasticity is the same for both kinds of stress, then the **neutral axis** of the section passes through its **center of gravity.**

The **elastic curve** is the curve assumed by a beam under load.

The **bending moment** for any section of a beam is the algebraic sum of the moments of the external or applied forces acting on the beam on one side of the section. Thus, for the beam shown, the bending moment about A is

$$M = R_1 x - Pa$$



The bending moment, M, of any section is numerically equal to the **moment of resistance** of the section, which is the resistance which the particles of the beam offer to distortion.

The **moment of resistance** equals

$$\frac{pI}{e} = M = \text{bending moment}$$

p = stress per unit area at the outermost element of the section,

e = distance of extreme element of beam from neutral axis,

I = rectangular moment of inertia of beam section about its horizontal gravity axis.

In designing the proper cross-section for a beam, the maximum bending moment (given for standard cases

under Beam Loadings) is equated to $\frac{pI}{e}$. The term $\frac{I}{e}$, called the section modulus, may be obtained from the table of standard sections of beams. The value of p must not exceed the maximum allowable stress per unit area for the material of the beam. The **maximum allowable stress** equals the ultimate strength divided by the factor of safety.

The **equation of the elastic curve** and its **radius of curvature** may be found from the relations:

$$M = \frac{pI}{e} = \frac{EI}{\rho} = EI\frac{d^2y}{dx^2} \text{ (approx.)}$$

E = modulus of elasticity of material of beam in tension or compression,

ρ = radius of curvature of the elastic curve,

(x, y) = coördinates of any point on the elastic curve.

The **deflection** of a beam at any point is obtained by substituting, in the equation of the elastic curve, the particular value of x in question, and solving for the corresponding value of y, which equals the deflection. The **maximum deflection** occurs at the section for which $\frac{dy}{dx} = 0$.

Shear

The **vertical shear** in a beam is equal to the first derivative of the bending moment in respect to x, thus

$$\text{Vertical shear} = J = \frac{dM}{dx}$$

where M is the bending moment (expressed as a function of x).

The value of the vertical shear for any particular

section is found by substituting the corresponding value of x in the expression for $\frac{dM}{dx}$. The result is the required vertical shear.

The **maximum bending moment** is found by equating $\frac{dM}{dx} = 0$, and then solving for the corresponding value of x. This particular value of x is substituted in the equation of the bending moment, M, and the resulting expression equals the maximum bending moment.

The **horizontal shear** in a plane parallel to the neutral surface (that is, the surface in which neither tension nor compression occurs), and at a distance z'' from it, equals

$$X \text{ (in pounds/sq. inch)} = \frac{J}{y''I}\int_{z''}^{e} z\,dF$$

where J = total vertical shear in pounds,
y'' = width of beam section at z'' in inches,
I = rectangular moment of inertia of entire section about the horizontal gravity axis,
$\int_{z''}^{e} z\,dF$ = area in square inches of that portion of the section above z'' multiplied by the distance in inches of its center of gravity above the neutral axis.

Beam Loadings

M = bending moment,
M_m = maximum bending moment,
y = deflection at any point,
d = maximum deflection,
P = concentrated load,
W = uniformly distributed load.

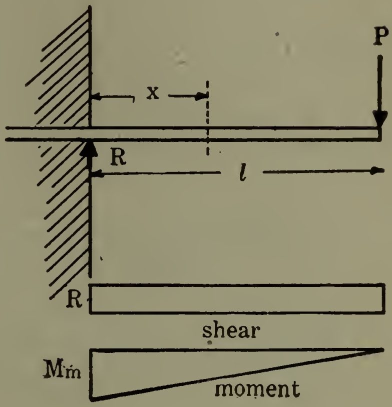
Cantilever Beam with Concentrated Load at the Free End

$$M = P(l - x)$$

$$M_m = Pl$$

$$y = \frac{P}{EI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right)$$

$$d = \frac{Pl^3}{3\,EI}$$



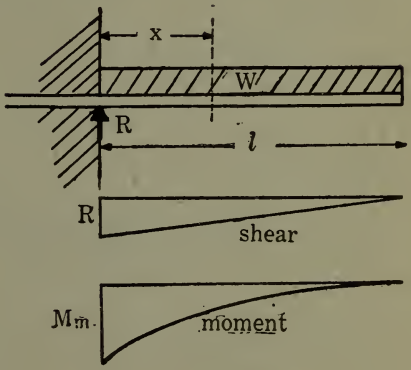
Cantilever Beam with Uniform Load

$$M = \frac{W(l - x)^2}{2\,l}$$

$$M_m = \frac{Wl}{2}$$

$$y = \frac{Wx^2\,[2\,l^2 + (2\,l - x)^2]}{24\,EIl}$$

$$d = \frac{Wl^3}{8\,EI}$$



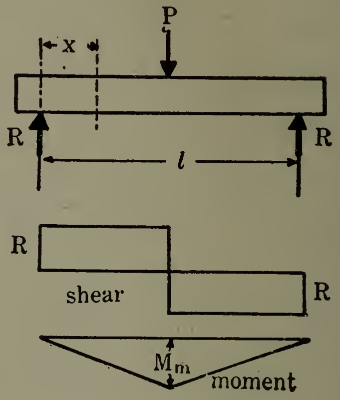
Beam Supported at Both Ends and Loaded with a Concentrated Load at Center

$$M = \frac{P}{2}x$$

$$M_m = \frac{Pl}{4}$$

$$y = \frac{Px\,(3\,l^2 - 4\,x^2)}{48\,EI}$$

$$d = \frac{Pl^3}{48\,EI}$$



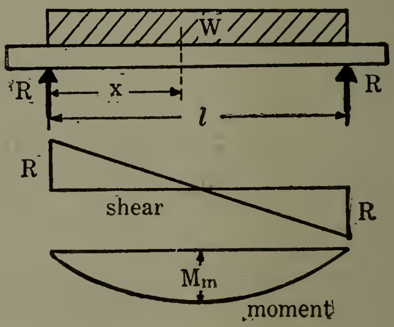
Beam Supported at Both Ends and Uniformly Loaded

$$M = \frac{Wx\,(l - x)}{2\,l}$$

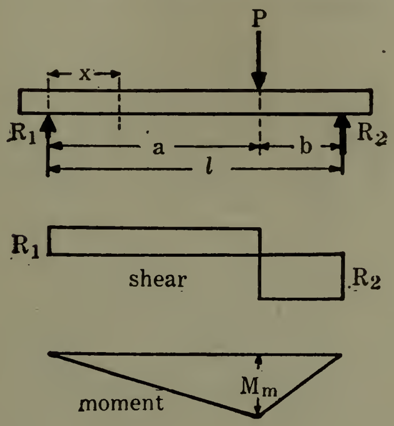
$$M_m = \frac{Wl}{8}$$

$$y = \frac{Wx\,(l^3 - 2\,lx^2 + x^3)}{24\,EIl}$$

$$d = \frac{5\,Wl^3}{384\,EI}$$



Beam Supported at Both Ends and Loaded at Any Point



$$M = \frac{Pbx}{l} \qquad x < a$$

$$M = \frac{Pbx}{l} - P(x - a) \qquad x > a$$

$$M_m = \frac{Pab}{l}$$

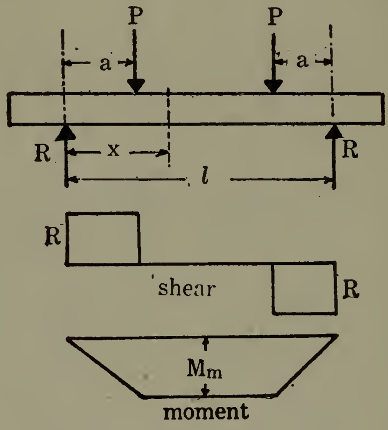
$$y = \frac{Pbx}{6\,EIl}(2\,la - a^2 - x^2) \qquad x < a$$

$$y = \frac{Pa\,(l - x)}{6\,EIl}(2\,lx - x^2 - a^2) \qquad x > a$$

$$d = \frac{Pb}{27\,EIl}\sqrt{3\,(2\,ab + a^2)^3}$$

occurring when $x = \frac{1}{3}\sqrt{3\,(2\,ab + a^2)}$

Beam Supported at Both Ends and Loaded with Two Concentrated Loads at Equal Distances from Each End



$$M = Px \qquad x < a$$

$$M = Pa \qquad x > a$$

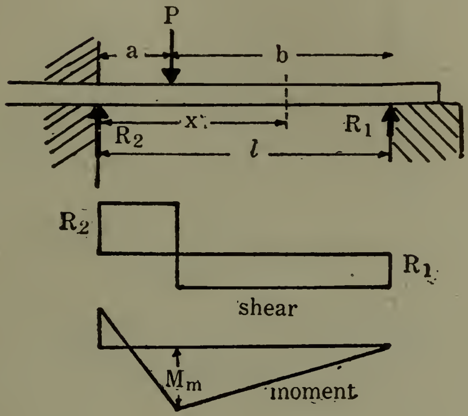
$$M_m = Pa$$

$$y = \frac{Px}{6\,EI}(3\,la - 3\,a^2 - x^2) \qquad x < a$$

$$y = \frac{Pa}{6\,EI}(3\,lx - 3\,x^2 - a^2) \qquad x > a$$

$$d = \frac{Pa}{6\,EI}\left(\frac{3}{4}l^2 - a^2\right)$$

Beam Fixed at One End, Supported at the Other, and with a Concentrated Load at Any Point



$$R_1 = \frac{Pa^2(3l-a)}{2l^3}$$

$$R_2 = P - R_1$$

$$M = P(a-x) - R_1(l-x) \qquad x < a$$

$$M = R_1(x-l) \qquad x > a$$

$$M_m = R_1(l-a)$$

$$y = \frac{1}{6EI}(R_1x^3 - 3R_1lx^2 + 3Pax^2 - Px^3) \qquad x < a$$

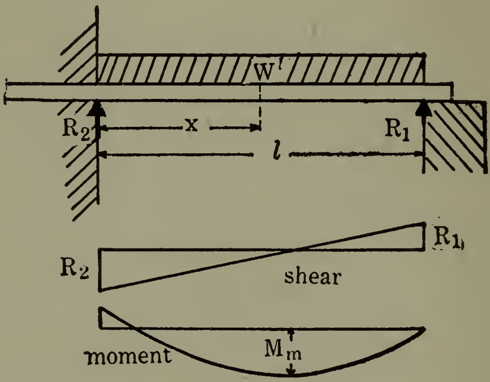
$$y = \frac{1}{6EI}(R_1x^3 - 3R_1lx^2 + 3Pa^2x - Pa^3) \qquad x > a$$

$$d = \frac{Pa^2}{6EI}(l-a)\sqrt{\frac{(l-a)}{(3l-a)}}$$

occurring when

$$x = l\left(1 - \sqrt{\frac{l-a}{3l-a}}\right)$$

Beam Fixed at One End, Supported at the Other and Uniformly Loaded



$$R_1 = \tfrac{3}{8} W$$

$$R_2 = \tfrac{5}{8} W$$

$$M = \frac{W}{8\,l}(l - 4\,x)\,(l - x)$$

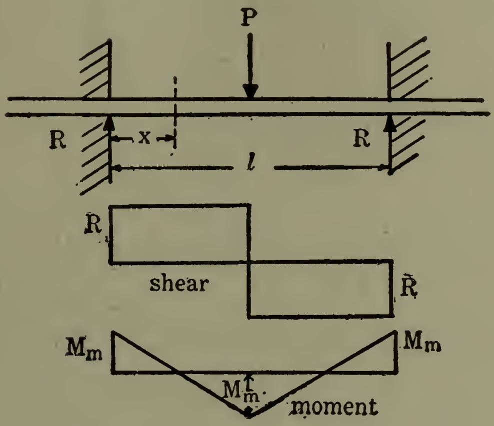
$$M_m = \frac{Wl}{8}$$

$$y = \frac{Wx^2}{48\,EIl}(l - x)\,(3\,l - 2\,x)$$

$$d = 0.0054\,\frac{Wl^3}{EI}$$

occurring when $x = 0.5785\,l$

Beam Fixed at Both Ends and Loaded at the Center



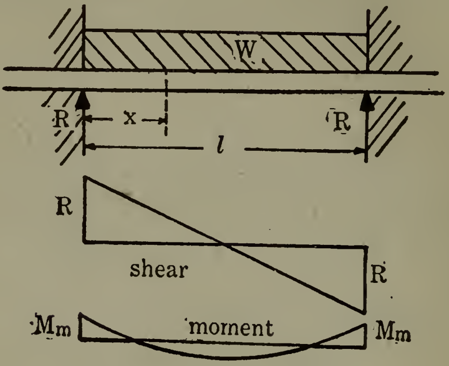
$$M = \frac{P}{8}(4x - l)$$

$$M_m = \frac{Pl}{8}$$

$$y = \frac{Px^2}{48\,EI}(4x - 3\,l)$$

$$d = \frac{Pl^3}{192\,EI}$$

Beam Fixed at Both Ends and Uniformly Loaded



$$M = \frac{W}{12\,l}(6\,lx - 6\,x^2 - l^2)$$

$$M_m = \frac{Wl}{12}$$

$$y = \frac{Wx^2}{24\,EIl}(l - x)^2$$

$$d = \frac{Wl^3}{384\,EI}$$

COLUMNS

Note. The breaking load in Euler's and in Gordon's formula, and the safe load in Ritter's formula are in pounds. In all of the formulæ for columns, the length, l, and radius of gyration, k, must be expressed in the same units (generally inches).

Euler's Formula

(1) Column with round ends,

$$\text{breaking load} = EI\frac{\pi^2}{l^2} = \pi^2 EF\left(\frac{k^2}{l^2}\right)$$

(2) Column with flat ends,

$$\text{breaking load} = 4\,EI\frac{\pi^2}{l^2} = 4\,\pi^2 EF\left(\frac{k^2}{l^2}\right)$$

(3) Pin-and-square column (column with one end round and the other flat),

$$\text{breaking load} = \frac{9}{4}\,EI\frac{\pi^2}{l^2} = \frac{9}{4}\pi^2\,EF\left(\frac{k^2}{l^2}\right)$$

in which

E = modulus of elasticity of material of column in tension or compression,
I = rectangular moment of inertia of cross-section about neutral axis,
l = length of column,
F = area of cross-section in sq. inches,
k = least radius of gyration of section.

Gordon's or Rankine's Formula

(1) Column with flat ends,

$$\text{breaking load} = \frac{FC}{1 + \beta\left(\frac{l}{k}\right)^2}$$

(2) Column with rounded ends,

$$\text{breaking load} = \frac{FC}{1 + 4\,\beta\left(\frac{l}{k}\right)^2}$$

(3) Pin-and-square column,

$$\text{breaking load} = \frac{FC}{1 + 1.78\,\beta\left(\frac{l}{k}\right)^2}$$

in which

F = area of cross-section in square inches,
C = ultimate compressive strength of material of column in pounds per square inch,
l = length of column,
k = least radius of gyration of section,
β = empirical constant.

Values of β and of C, in Gordon's formula, are as follows for different materials:

Material	Hard steel	Medium steel	Soft steel	Wrought iron	Cast iron	Timber
C (lbs./sq. in.).	70,000	50,000	45,000	36,000	70,000	7200
β........	$\frac{1}{25,000}$	$\frac{1}{36,000}$	$\frac{1}{36,000}$	$\frac{1}{36,000}$	$\frac{1}{6400}$	$\frac{1}{3000}$

Ritter's Formula

(1) Column with flat ends,

$$\text{safe load} = \frac{FC}{1 + \frac{C'}{4\pi^2 E}\left(\frac{l}{k}\right)^2}$$

(2) Column with rounded ends,

$$\text{safe load} = \frac{FC}{1 + \frac{C'}{\pi^2 E}\left(\frac{l}{k}\right)^2}$$

(3) Pin-and-square column,

$$\text{safe load} = \frac{FC}{1 + \frac{1.78\,C'}{4\pi^2 E}\left(\frac{l}{k}\right)^2}$$

in which

F = area of cross-section in square inches,
C = maximum safe compressive stress of material of column in pounds per square inch,
C' = compressive stress at elastic limit in pounds per square inch,
E = modulus of elasticity for tension or compression,
l = length of column,
k = least radius of gyration.

J. B. Johnson's Formula

Breaking load in pounds; cross-section in square inches.

For **mild steel:**

(1) Pin-ends,

$$\text{breaking load} = \left[42{,}000 - 0.97\left(\frac{l}{k}\right)^2\right]F$$

$$\left(\frac{l}{k}\right) \text{ not} > 150$$

(2) Flat ends,

$$\text{breaking load} = \left[42{,}000 - 0.62\left(\frac{l}{k}\right)^2\right]F$$

$$\left(\frac{l}{k}\right) \text{ not} > 190$$

For **wrought iron:**

(1) Pin-ends,

$$\text{breaking load} = \left[34{,}000 - 0.67\left(\frac{l}{k}\right)^2\right]F$$

$$\left(\frac{l}{k}\right) \text{ not} > 170$$

(2) Flat ends,

$$\text{breaking load} = \left[34{,}000 - 0.43\left(\frac{l}{k}\right)^2\right]F$$

$$\left(\frac{l}{k}\right) \text{not} > 210$$

Notation same as in Ritter's formula.

Straight-line Formula

Breaking load in pounds; cross-section in square inches.

For **mild steel:**

(1) Hinged ends,

$$\text{breaking load} = \left[52{,}000 - 220\left(\frac{l}{k}\right)\right]F$$

(2) Flat ends,

$$\text{breaking load} = \left[52{,}000 - 179\left(\frac{l}{k}\right)\right]F$$

For **wrought iron:**

(1) Hinged ends,

$$\text{breaking load} = \left[42{,}000 - 157\left(\frac{l}{k}\right)\right]F$$

(2) Flat ends,

$$\text{breaking load} = \left[42{,}000 - 128\left(\frac{l}{k}\right)\right]F$$

Notation same as in Ritter's formula.

Wooden Columns

The breaking load in pounds for solid wooden columns with square ends is

$$P = \frac{(700 + 15\,m)\,FC}{700 + 15\,m + m^2}$$

F = cross-section in square inches,

m = ratio of the length, l, of the column to the least dimension d, of the cross-section $\left(\text{that is, } m = \frac{l}{d}\right)$,

C = ultimate compressive strength of material of column in pounds per square inch.

Values of C, the ultimate compressive strength, for different kinds of timber are as follows:

White oak and Georgia yellow pine.....	5000 lb./sq. in.
Douglas fir and short-leaf yellow pine...	4500 lb./sq. in.
Red pine, spruce, hemlock, cypress, chestnut, California redwood, and California spruce....................	4000 lb./sq. in.
White pine and cedar................	3500 lb./sq. in.

The **proper factor of safety** for **yellow pine** varies from 3.5 to 5, according to the amount of moisture present in the timber, being greater for larger amounts of moisture. For all other timbers, the proper factor of safety varies from 4 to 5.

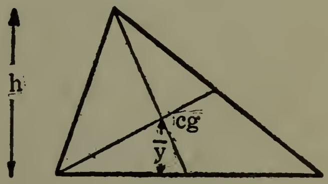
CENTERS OF GRAVITY

Plane Figures

Triangle

The C.G. is on a median line of the triangle, two-thirds of its length from the vertex,

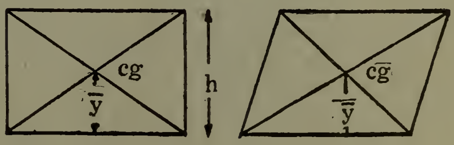
$$\bar{y} = \frac{h}{3}$$



Parallelogram

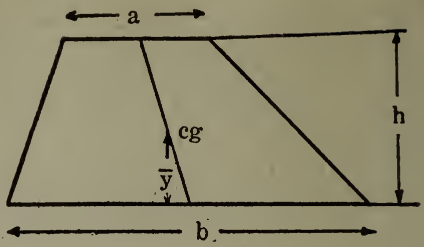
The C.G. is at the intersection of the diagonals,

$$\bar{y} = \frac{h}{2}$$



Trapezoid

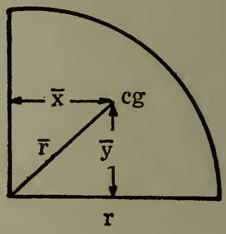
$$\bar{y} = \frac{h\,(2\,a + b)}{3\,(a + b)}$$



Quadrant of Circle

$$\bar{x} = \frac{4\,r}{3\,\pi} = \bar{y}$$

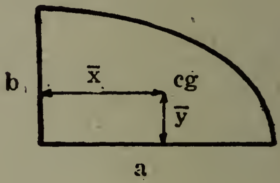
$$\bar{r} = \frac{4\,r\,\sqrt{2}}{3\,\pi}$$



Quadrant of Ellipse

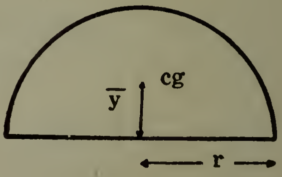
$$\bar{x} = \frac{4\,a}{3\,\pi}$$

$$\bar{y} = \frac{4\,b}{3\,\pi}$$



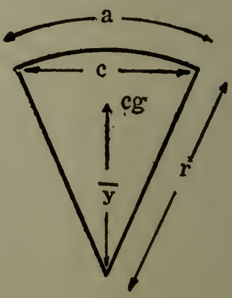
Semicircle

$$\bar{y} = \frac{4\,r}{3\,\pi}$$



Circular Sector

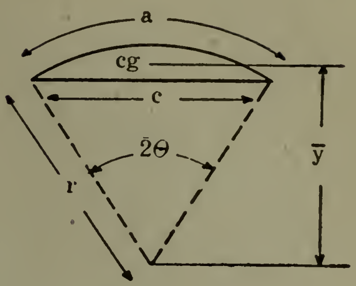
$$\bar{y} = \frac{2\,rc}{3\,a}$$



Circular Segment

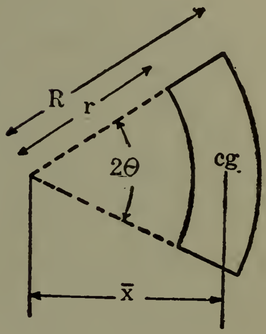
$$\bar{y} = \frac{4}{3}\frac{r\sin^3\theta}{2\theta - \sin(2\theta)}$$

θ is in radians



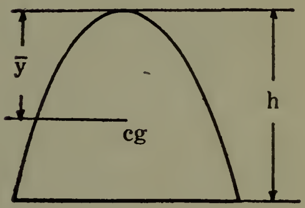
Sector of a Circular Ring

$$\bar{x} = \frac{2}{3}\frac{R^3 - r^3}{R^2 - r^2}\frac{\sin\theta}{\theta}$$



Parabolic Segment

$$\bar{y} = \frac{3h}{5}$$



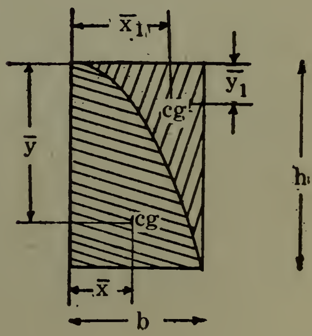
Parabolic Segment

$\bar{x} = \frac{3}{8}b$

$\bar{y} = \frac{3}{5}h$

$\bar{x}_1 = \frac{3}{4}b$

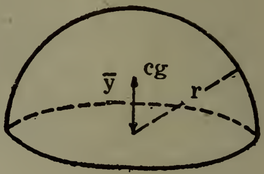
$\bar{y}_1 = \frac{3}{10}h$



Solids

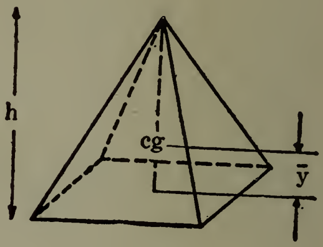
Hemisphere

$$\bar{y} = \frac{3r}{8}$$



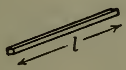
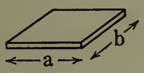
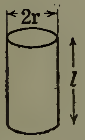
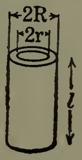
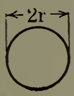
Right Pyramid or Cone

$$\bar{y} = \frac{h}{4}$$

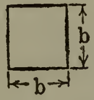
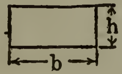
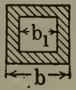
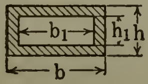


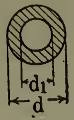
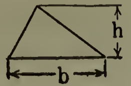
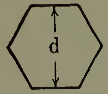
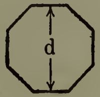
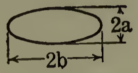
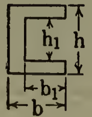
MOMENT OF INERTIA OF SOLIDS

$$M = \text{mass of body} = \frac{W}{g}$$

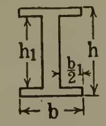
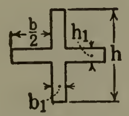
Shape of figure	Description	Axis of rotation	Moment of inertia
	uniform thin rod	(1) through center perpendicular to length (2) through end perpendicular to length	$M\frac{l^2}{12}$ $M\frac{l^2}{3}$
	thin rectangular plate	(1) through center of gravity perpendicular to plate (2) through center of gravity parallel to side b	$M\frac{a^2+b^2}{12}$ $M\frac{a^2}{12}$
	thin circular plate	(1) through center perpendicular to plane (2) any diameter	$M\frac{r^2}{2}$ $M\frac{r^2}{4}$
	solid cylinder, radius, r	(1) axis of cylinder (2) through center of gravity, perpendicular to axis of cylinder	$M\frac{r^2}{2}$ $M\left(\frac{l^2}{12}+\frac{r^2}{4}\right)$
	hollow cylinder, R=outer radius, r=inner radius	(1) axis of cylinder (2) through center of gravity perpendicular to axis	$M\left(\frac{R^2+r^2}{2}\right)$ $M\left(\frac{l^2}{12}+\frac{R^2+r^2}{4}\right)$
	solid sphere, r=radius	through center	$M\frac{2r^2}{5}$
	hollow sphere, R=external radius r=internal radius	through center	$M\frac{2}{5}\left(\frac{R^5-r^5}{R^3-r^3}\right)$

PROPERTIES OF STANDARD SECTIONS

Shape of section	Area of section	Rectangular moment of inertia about horizontal gravity axis I	Square of radius of gyration $k^2=\frac{I}{A}$	Section modulus $s=\frac{I}{e}$	Polar moment of inertia, I_p about center of gravity
	b^2	$\frac{b^4}{12}$	$\frac{b^2}{12}$	$\frac{b^3}{6}$	$\frac{b^4}{6}$
	bh	$\frac{bh^3}{12}$	$\frac{h^2}{12}$	$\frac{bh^2}{6}$	$\frac{bh}{12}(b^2+h^2)$
	$b^2-b_1^2$	$\frac{b^4-b_1^4}{12}$	$\frac{b^2+b_1^2}{12}$	$\frac{b^4-b_1^4}{6\,b}$	$\frac{b^4-b_1^4}{\ }$
	$bh-b_1h_1$	$\frac{bh^3-b_1h_1^3}{12}$	$\frac{bh^3-b_1h_1^3}{12\,(bh-b_1h_1)}$	$\frac{bh^3-b_1h_1^3}{6\,h}$	$\frac{bh\,(h^2+b^2)}{12}-\frac{b_1h_1\,(h_1^2+b_1^2)}{12}$
	$\frac{\pi d^2}{4}$	$\frac{\pi d^4}{64}$	$\frac{d^2}{16}$	$\frac{\pi d^3}{32}$	$\frac{\pi d^4}{32}$

	$\frac{\pi (d^2-d_1^2)}{4}$	$\frac{\pi (d^4-d_1^4)}{64}$	$\frac{d^2+d_1^2}{16}$	$\frac{\pi (d^4-d_1^4)}{32\,d}$	$\frac{\pi (d^4-d_1^4)}{32}$
	$\frac{bh}{2}$	$\frac{bh^3}{36}$	$\frac{h^2}{18}$	$\frac{bh^2}{24}$	
	$0.866\ d^2$	$0.06\ d^4$	$0.0697\ d^2$	$0.12\ d^3$	
	$0.828\ d^2$	$0.055\ d^4$	$0.066\ d^2$	$0.109\ d^3$	
	πab	$\frac{\pi ba^3}{4}$	$\frac{a^2}{4}$	$\pi \frac{ba^2}{4}$	$\frac{\pi}{4}(a^3b+b^3a)$
	$bh-b_1h_1$	$\frac{bh^3-b_1h_1^3}{12}$	$\frac{1}{12}\left(\frac{bh^3-b_1h_1^3}{bh-b_1h_1}\right)$	$\frac{bh^3-b_1h_1^3}{6\,h}$	

PROPERTIES OF STANDARD SECTIONS—*Continued*

	$bh-b_1h_1$	$\dfrac{bh^3-b_1h_1^3}{12}$	$\dfrac{1}{12}\left(\dfrac{bh^3-b_1h_1^3}{bh-b_1h_1}\right)$	$\dfrac{bh^3-b_1h_1^3}{6\,h}$	
	b_1h+bh_1	$\dfrac{b_1h^3+bh_1^3}{12}$	$\dfrac{1}{12}\left(\dfrac{b_1h^3+bh_1^3}{b_1h+bh_1}\right)$	$\dfrac{b_1h^3+bh_1^3}{6\,h}$	

The moment of inertia of such sections as T-beams and angle-bars, the center of gravity of which cannot be determined by inspection, may be obtained as follows: First, find the position of the horizontal gravity axis by the method given on page 74, for Composite Sections. Then divide the section into its component rectangles, with their bases along the gravity axis. The moment of inertia of each rectangle about its base is calculated by the formula $I = \dfrac{bh^3}{3}$ where b is the base of the rectangle and h its altitude.

The total moment of inertia of the section about its gravity axis is the sum of the moments of inertia of the component rectangles. If there is a rectangular space in the figure, the corresponding moment of inertia is subtracted from that of the solid section.

HYDRAULICS

Head and Pressure

The difference in level of water between two points is called the **head.**

The **pressure** in pounds per square inch at any depth is

$$p = 0.433\ h$$

in which

h = head or depth in feet of water,
0.433 = weight of a column of water 1 foot high and 1 inch in cross-section.

The pressure on a **submerged surface** is always normal to the surface, and equals

$$P \text{ (in pounds)} = 0.433\ hF$$

h = depth of water in feet from the surface of the liquid to the center of gravity of the submerged surface,
F = area of submerged surface in square inches.

Center of Pressure

The **center of pressure** of a submerged surface is the point of application of the resultant of all the fluid pressures on such surface.

The distance of the center of pressure of a **vertical submerged plate** below the liquid surface is

$$d \text{ (in feet)} = \frac{I_s}{F\bar{z}}$$

F = area of plate in square feet,
$\bar{z}$ = distance in feet from the liquid surface to the center of gravity of the plate,
I_s = rectangular moment of inertia of plate about the line of intersection of its plane with the surface of the liquid.

The distance of the center of pressure of a **submerged plate inclined** at an angle θ with the surface is

$$d \text{ (in feet)} = \frac{I_g \sin^2 \theta}{F\bar{z}} + \bar{z}$$

$\bar{z}$ = distance from the liquid surface to the center of gravity of the plate in feet,
F = area of plate in square feet,
I_g = moment of inertia of plate about its gravity axis parallel to the liquid surface.

Flow through Apertures

Due to friction, the velocity of discharge through an aperture in a thin plate or plank is reduced about 3 per cent below its theoretical value. Further, on leaving the orifice, the jet contracts to approximately 64 per cent of the area of the aperture.

The **theoretical** velocity of discharge through a small aperture, in feet per second, is

$$v = \sqrt{2\,gh}$$

g = acceleration of gravity = **32.16,**
h = head in feet.

The **actual** velocity of discharge in feet per second is

$$v = \phi\sqrt{2\,gh} = 0.97\sqrt{2\,gh}$$

ϕ = coefficient of velocity.

The **discharge** through the aperture in cubic feet per second is

$$Q = CF\phi\sqrt{2\,gh} = 0.62\,F\sqrt{2\,gh}$$

C = 0.64 (approx.) = coefficient of contraction,
F = area of aperture in square feet.

FLOW OF WATER IN PIPES

Bernoulli's Theorem

A general method for calculating the flow of water in pipes is given by Bernoulli's theorem:

$$\frac{v^2}{2\,g} + \frac{p}{\gamma} + \bar{z} = \frac{v_1^2}{2\,g} + \frac{p_1}{\gamma} + \bar{z}_1 + k$$

that is, the sum of the velocity head $\frac{v^2}{2\,g}$, the pressure head $\frac{p}{\gamma}$ and the potential head $\bar{z}$ at any given section of flow is equal to the sum of the corresponding heads at any other section, plus the various losses between the two sections considered.

v = velocity in feet per second at first section,
v_1 = velocity at second section,
p = pressure in pounds per square inch at first section,
p_1 = pressure at second section,

$\bar{z}$ = potential head at first section in feet, that is, the distance of the center of the section above a chosen horizontal reference plane,
$\bar{z}_1$ = potential head at second section,
g = 32.16 (approx.),
γ = weight in pounds of a column of water 1 foot high and 1 square inch in cross-section = 0.433,
k = various losses in feet of head between the two sections of pipe considered.

Losses in Pipes

The following formulæ for losses in pipes enable us to find the value of the term k appearing in Bernoulli's theorem. If several losses occur in a section of pipe, the total loss, k, is the sum of the separate losses.

Loss Due to Friction

The loss of head in feet due to friction in a section of pipe is

$$4f\frac{l}{d}\frac{v^2}{2g}$$

where

d = diameter of pipe in feet,
l = length of pipe in feet,
v = velocity in feet per second,
f = coefficient of friction, depending on the velocity, and on the size of pipe.

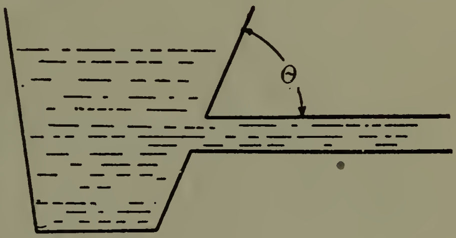
Values of f, the coefficient of friction, for water in clean iron pipes are as follows (condensed from I. P. Church's "Mechanics of Engineering"):

Velocity in feet per second	Diam. = ½ in. = 0.0417 ft.	Diam. = 1 in. = 0.0834 ft.	Diam. = 2 in. = 0.1667 ft.	Diam. = 4 in. = 0.333 ft.	Diam. = 8 in. = 0.667 ft.	Diam. = 12 in. = 1.00 ft.	Diam. = 16 in. = 1.333 ft.	Diam. = 20 in. = 1.667 ft.
0.1	0.0150	0.0119	0.00870	0.00763	0.00704	0.00669	0.00623	......
0.3	0.0137	0.0113	0.00850	0.00750	0.00693	0.00657	0.00614	0.00578
0.6	0.0124	0.0104	0.00822	0.00732	0.00677	0.00642	0.00603	0.00567
1.0	0.0110	0.00950	0.00790	0.00712	0.00659	0.00624	0.00588	0.00555
2.0	0.00862	0.00810	0.00731	0.00678	0.00624	0.00593	0.00559	0.00529
3.0	0.00753	0.00734	0.00692	0.00650	0.00600	0.00570	0.00538	0.00509
6.0	0.00689	0.00670	0.00640	0.00605	0.00562	0.00534	0.00507	0.00482
12.0	0.00630	0.00614	0.00590	0.00560	0.00522	0.00500	0.00478	0.00457
20.0	0.00615	0.00598	0.00579	0.00549	0.00508	0.00485	......	......

Loss at Entrance

The loss of head in feet due to entrance from a reservoir into a pipe is equal to

$$\left(\frac{1}{\phi^2} - 1\right)\frac{v^2}{2g} = L_e\frac{v^2}{2g}$$



in which ϕ is the coefficient of friction and is dependent on the angle $\theta°$ which the pipe makes with the inner surface of the reservoir.

Values of $L_e\left(= \frac{1}{\phi^2} - 1\right)$ in the above formula are as follows for different values of $\theta°$ (from Church):

$\theta°$	90°	80°	70°	60°	50°	40°	30°
L_e	0.505	0.565	0.635	0.713	0.794	0.870	0.987

Thus, when the discharge is through a pipe normal to the inner surface of the reservoir, then $\theta°$ equals 90° and L_e is, therefore, 0.505, the loss at entrance then being

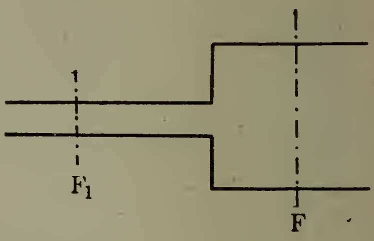
$$0.505\frac{v^2}{2g}$$

where v = velocity of flow in pipe in feet per second.

Loss Due to Sudden Enlargement

The loss of head in feet due to the sudden enlargement of a pipe is

$$\left(\frac{F}{F_1} - 1\right)^2 \frac{v^2}{2g}$$



F_1 = cross-section area of the smaller pipe in square feet,

F = area of enlarged section in square feet,

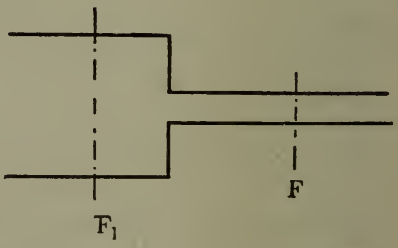
v = velocity in feet per second in the enlarged section.

Loss Due to Sudden Contraction

The loss of head in feet due to the sudden contraction of a pipe is

$$\left(\frac{1}{C} - 1\right)^2 \frac{v^2}{2g}$$

in which



v = velocity in feet per second in contracted section,

C = coefficient of contraction, the value of which depends on the ratio, $\frac{F}{F_1}$, of the small section to the large section.

Values of C, the coefficient of contraction, for

different values of $\frac{F}{F_1}$ are given in the following table (from Church):

$\frac{F}{F_1}$	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.0
C	0.624	0.632	0.643	0.659	0.681	0.712	0.755	0.813	0.892	1.0

Loss Due to Bends

The loss of head in feet due to a bend in a circular pipe is

$$\left[0.131 + 1.847\left(\frac{a}{r}\right)^{\frac{7}{2}}\right]\frac{v^2}{2g} = L_b\frac{v^2}{2g}$$

a = radius of pipe in feet,
r = radius of bend in feet,
v = velocity of flow in feet per second.

Values of L_b for different values of $\frac{a}{r}$ are as follows:

$\frac{a}{r}$	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
L_b	0.131	0.138	0.158	0.206	0.294	0.440	0.661	0.977	1.40	1.98

Flow Through Straight Cylindrical Pipes

Q = discharge in cubic feet per second,
v = velocity of discharge in feet per second,
l = length of pipe in feet,
d = diameter of pipe in feet,
L_e = coefficient of loss at entrance. In general, the pipe is normal to the inner surface of the reservoir and then $L_e = 0.505$. For other cases see Loss at Entrance.
f = coefficient of friction, obtained from the table on page 123.

(1) Required the head in feet necessary to keep up a given flow of Q cubic feet per second in a clean iron pipe of given length l and diameter d.

The required head is

$$h \text{ (in feet)} = \frac{v^2}{2g}\left(1 + L_e + 4f\frac{l}{d}\right)$$

in which
$$v = \frac{4Q}{\pi d^2}$$

(2) Required the velocity in the pipe, having given the head h and the length l and the diameter d of the pipe; also required the discharge Q in cubic feet per second.

The velocity in feet per second is:

$$v = \sqrt{\frac{2gh}{1 + L_e + 4f\frac{l}{d}}}$$

and after solving for v,

$$Q = \tfrac{1}{4}\pi d^2 v$$

Since the value of f depends on the unknown v as well as the known d, we may first put $f = 0.006$ for a trial approximation and solve for v; then take the value of f corresponding to this velocity and substitute again in the given formula for v. One trial is generally sufficient for ordinary accuracy.

(3) Required the proper diameter d for the pipe to discharge a given quantity Q cubic feet per second, having given the length of pipe and the head h.

The proper diameter in feet is

$$d = \sqrt[5]{\frac{(1 + L_e)\,d + 4fl}{2gh}\left(\frac{4Q}{\pi}\right)^2}$$

and d being solved for,

$$v = \frac{4\,Q}{\pi d^2}$$

Since the radical contains d, we must first assume a trial value for d, and taking $f = 0.006$, substitute in the above formula for the diameter. Having obtained a value for d, we solve for the velocity v. With the approximate values of d and v thus obtained, we find the corresponding new value of f from the table of friction, and then substitute again in the formulæ. One or two trials generally give sufficient accuracy.

Flow Through Very Long Pipes

When a pipe is very long (1000 feet or more), the head, velocity, or discharge, etc., may be calculated from the formulæ:

$$h = 4f\frac{l}{d}\,\frac{v^2}{2\,g} \qquad \text{(Chézy's formula)}$$

$$v = \frac{4\,Q}{\pi d^2}$$

$$h = \frac{64}{\pi^2}f\frac{lQ^2}{d^5 2\,g}$$

Notation same as in preceding section.

FLOW THROUGH OPEN CHANNELS

Bazin's Formula

The velocity of flow in a channel in feet per second is

$$v = \frac{87\sqrt{rs}}{0.552 + \dfrac{m}{\sqrt{r}}}$$

r = mean hydraulic radius in feet, which is found by dividing the area of the fluid cross-section in square feet by the wetted perimeter in feet (that is, the perimeter of the channel section in contact with the water),

s = slope of stream (that is, the difference in elevation between two points of the water surface divided by the distance between the two points measured along the surface),

m = coefficient of roughness, the values of which are given in the following table.

Character of channel	Value of m
Very smooth cement surfaces or planed boards..	0.06
Concrete, well-laid brick, unplaned boards......	0.16
Ashlar, good rubble masonry, poor brickwork...	0.46
Earth beds in perfect condition.................	0.85
Earth beds in ordinary condition...............	1.30
Earth beds in bad condition covered with débris	1.75

Kutter's Formula

The velocity of flow in a channel in feet per second equals

$$v = \frac{41.65 + \dfrac{0.00281}{s} + \dfrac{1.811}{n}}{1 + \left(41.65 + \dfrac{0.00281}{s}\right)\dfrac{n}{\sqrt{r}}}\sqrt{rs}$$

where r and s are as in Bazin's formula.

Values for n, the coefficient of roughness, are as follows:

Character of channel	Value of n
Planed timber, glazed or enameled surfaces....	0.009
Smooth clean cement..........................	0.010
Unplaned timber, new well-laid brickwork.....	0.012
Smooth stonework, ordinary brickwork, iron...	0.013
Rough ashlar and good rubble masonry........	0.017
Firm gravel..................................	0.020
Earth in ordinary condition..................	0.025
Earth with stones, weeds, etc................	0.030
Earth or gravel in bad condition.............	0.035

FLOW OVER WEIRS

Contraction is **complete** when no edge of the weir is flush with the sides or bottom of the channel.

Contraction is **incomplete** when one or more sides of the weir have an interior border flush with the sides or bottom of the channel.

Francis' Formula

The flow over a weir in cubic feet per second is

$$Q = \tfrac{2}{3}\left[0.622\, h\left(b - \tfrac{1}{10}\, nh\right)\sqrt{2\, gh}\right]$$

in which

h = head in feet of water on weir,

b = width of weir in feet,

n = 2 for complete contraction,

n = 1 for one end of weir flush with side of channel,

n = 0 for both ends of weir flush with sides of channel.

Bazin's Formula for Weirs

For overfall-weirs with end contractions suppressed, the flow in cubic feet per second is

$$Q = \frac{2}{3} n\left[1 + 0.55\left(\frac{h}{p+h}\right)^2\right] bh\sqrt{2\, gh}$$

in which the coefficient n has the value

$$n = 0.6075 + \frac{0.0148}{h}$$

h = depth in feet of water on weir,
b = width of weir in feet,
p = height in feet of the sill of the weir above the bottom of the channel of approach.

STRESSES IN PIPES AND CYLINDERS

Pressure in Pipes

The tensile stress in pounds per square inch in a pipe due to internal fluid pressure is:

For **thin pipes**, $p' = \frac{rp}{t}$

For **thick pipes or cylinders**,

$$p' = \frac{p\,(r + t)}{t}$$

r = inside radius of pipe in inches,
t = thickness of pipe in inches,
p = excess of internal over external pressure in pounds per square inch.

If S is the required factor of safety, then:

For **thin pipes**, $t = S\frac{rp}{P}$

For **thick pipes or cylinders**,

$$t = S\frac{rp}{P - pS}$$

in which r and p are as above, and

P = ultimate tensile strength of material of pipe (see Table of Strength of Materials).

Collapsing of Tubes

The collapsing pressure for Bessemer steel lap-welded tubes, for lengths greater than six diameters, is

$$\left.\begin{aligned} p &= 1000\left(1 - \sqrt{1 - 1600\frac{t^2}{d^2}}\right) \quad \text{when } \frac{t}{d} < 0.023 \\ \text{or} \\ p &= 86670\frac{t}{d} - 1386 \qquad\qquad \text{when } \frac{t}{d} > 0.023 \end{aligned}\right\}$$

(Stewart's equations)

in which

p = excess of external over internal pressure in pounds per square inch,
d = outside diameter of tube in inches,
t = thickness of tube wall in inches.

FLOW OF FLUIDS

Flow of Air Through Apertures

The weight of air in pounds discharged per second from a reservoir into the atmosphere is

$$\left.\begin{aligned} M &= 0.53\,F\frac{p_1}{\sqrt{T_1}} \quad \text{when} \quad p_1 > 2\,p_a \\ \text{or} \\ M &= 1.06\,F\sqrt{\frac{p_a(p_1 - p_a)}{T_1}} \quad \text{when} \quad p_1 < 2\,p_a \end{aligned}\right\} \text{Fliegner's equations}$$

p_1 = reservoir pressure in pounds per square inch absolute,
p_a = atmospheric pressure (14.7 pounds per square inch),
F = cross-section of aperture in square inches,
T_1 = absolute temperature of reservoir (degrees Fahr. + 459.6).

Flow of Steam Through Apertures

$$M = 0.0165\, Fp_1^{0.97} \qquad \text{(Grashof's formula)}$$

$$\left.\begin{array}{l} M = \dfrac{Fp_1}{70} \quad \text{when} \quad p_1 > \dfrac{5}{3}p_2 \\ M = \dfrac{Fp_2}{42}\sqrt{\dfrac{3\,(p_1 - p_2)}{2\,p_2}} \quad \text{when} \quad p_1 < \dfrac{5}{3}p_2 \end{array}\right\} \text{Napier's equations}$$

Grashof's formula applies when the final pressure is less than 58 per cent of the reservoir pressure.

M = pounds of steam discharged per second,
p_1 = reservoir pressure in pounds per square inch,
p_2 = final pressure in pounds per square inch,
F = cross-section of aperture in square inches.

Flow of Gas in Pipes

$$Q = 1000\sqrt{\frac{d^5h}{sl}} \qquad \text{(Molesworth)}$$

Q = quantity of gas in cubic feet per hour,
d = diameter of pipe in inches,
l = length of pipe in yards,
h = pressure in inches of water,
s = specific gravity of gas relative to air.

Flow of Air in Pipes

$$v = 114.5\sqrt{\frac{hd}{L}} \qquad \text{(Hawksley)}$$

v = velocity in feet per second,
h = head in inches of water,
d = diameter of pipe in inches,

L = length of pipe in feet,

$$Q = \frac{\pi}{4}\frac{d^2}{144}v$$

Q = quantity in cubic feet per second.

Flow of Compressed Air in Pipes

$$Q = 217.5\sqrt{\frac{pd^5}{rL}}$$

$$d = 0.1161\sqrt[5]{\frac{LQ^2r}{p}} = 0.1161\sqrt[5]{\frac{LQ_1^2}{pr}}$$

Q = volume in cubic feet per minute of compressed air, at 62° F.,
Q_1 = volume before compression, at 62° F.,
r = pressure in atmospheres,
p = difference in pressures in pounds per sq. inch, causing the flow,
d = diameter of pipe in inches,
L = length of pipe in feet.

Flow of Steam in Pipes

$$W = 87\sqrt{\frac{w\,(p_1 - p_2)\,d^5}{L\left(1 + \frac{3.6}{d}\right)}} \qquad \text{(Babcock)}$$

W = weight of steam flowing in pounds per minute,
w = density in pounds per cubic foot of the steam at the entrance to the pipe,
p_1 = pressure in pounds per square inch at the entrance,
p_2 = pressure at exit,
d = diameter in inches,
L = length of pipe in feet.

ELECTRICITY

OHMIC RESISTANCE

The resistance of a uniform electric conductor at 0° Centigrade is given by the formula:

$$R \text{ (in ohms)} = \rho \frac{L}{A}$$

L = length of conductor in inches,
A = cross-section in square inches,
ρ = resistivity of conductor at 0° C., values of which are given in the following table.

TABLE OF RESISTIVITIES

(Resistivity is the resistance in ohms between any two opposite faces of a 1 inch cube of the material)*

Metal	Resistivity at 0° C.
Aluminium (annealed)	1.14×10^{-6}
Aluminium (commercial)	1.05×10^{-6}
Aluminium bronze	4.96×10^{-6}
Bismuth (compressed)	51.2×10^{-6}
Brass	2.82×10^{-6}
Copper (drawn)	0.637×10^{-6}
Copper (annealed)	0.625×10^{-6}
German silver	8.23×10^{-6}
Gold (annealed)	0.803×10^{-6}
Iron (wrought)	3.82×10^{-6}
Lead (compressed)	7.68×10^{-6}
Magnesium	1.72×10^{-6}
Mercury	37.1×10^{-6}
Nickel (annealed)	4.89×10^{-6}
Platinum (annealed)	3.53×10^{-6}
Silver (annealed)	0.575×10^{-6}
Tin	5.16×10^{-6}
Tungsten	$2. \times 10^{-6}$
Zinc (pressed)	2.28×10^{-6}

* This definition applies to English units and to the numerical values given in the table. In general, resistivity is the resistance of a unit cube.

The **resistance** of a conductor at **any temperature** is

$$R_2 = R_1 \frac{(1 + \alpha t_2)}{(1 + \alpha t_1)}$$

in which

R_1 = known resistance at a temperature t_1 degrees Centigrade,

R_2 = required resistance at a temperature t_2 degrees Centigrade,

α = temperature coefficient of electrical resistance, the value of which is given for different metals in the following table.

TEMPERATURE COEFFICIENTS OF ELECTRICAL RESISTANCE

Metal	Temp. coefficient (approx.) for 1° C.
Aluminium (commercial)	0.00435
Copper (annealed)	0.00388
German silver	0.00036
Gold (annealed)	0.00365
Iron (wrought)	0.00463
Mercury	0.00072
Platinum	0.00247
Silver	0.00377
Tungsten	0.00570
Zinc (pressed)	0.00365

Note.—The temperature coefficient of a material is its increase in resistance for each degree Centigrade rise in temperature, and it is expressed as a decimal fraction of the resistance at 0° C.

DATA ON ANNEALED COPPER WIRE

Gauge No. (B. & S.)	Diameter in mils			Cross-section of bare wire		Resistance in ohms per 1000 feet		Pounds per 1000 feet
	Bare	Single cotton covered	Double cotton covered	Circular mils	Sq. inches	Cold (25° C. = 77° F.)	Hot (65° C. = 149° F.)	
0000	460	....	....	212,000	0.166	0.0500	0.0577	641
000	410	....	....	168,000	0.132	0.0630	0.0727	508
00	365	....	....	133,000	0.105	0.0795	0.0917	403
0	325	....	....	106,000	0.0829	0.100	0.116	319
1	289	....	....	83,700	0.0657	0.126	0.146	253
2	258	....	....	66,400	0.0521	0.159	0.184	201
3	229	....	....	52,600	0.0413	0.201	0.232	159
4	204	211	....	41,700	0.0328	0.253	0.292	126
5	182	189	....	33,100	0.0260	0.319	0.369	100
6	162	169	174	26,300	0.0206	0.403	0.465	79.5
7	144	151	156	20,800	0.0164	0.508	0.586	63.0
8	128	136	141	16,500	0.0130	0.641	0.739	50.0
9	114	121	126	13,100	0.0103	0.808	0.932	39.6
10	102	108	112	10,400	0.00815	1.02	1.18	31.4
11	91	97	101	8,230	0.00647	1.28	1.48	24.9
12	81	87	91	6,530	0.00513	1.62	1.87	19.8
13	72	78	82	5,180	0.00407	2.04	2.36	15.7
14	64	70	74	4,110	0.00323	2.58	2.97	12.4
15	57	63	67	3,260	0.00256	3.25	3.75	9.86
16	51	56	59	2,580	0.00203	4.09	4.73	7.82
17	45	50	53	2,050	0.00161	5.16	5.96	6.20

DATA ON ANNEALED COPPER WIRE (*Continued*)

Gauge No. (B. & S.)	Diameter in mils			Cross-section of bare wire		Resistance in ohms per 1000 ft.		Pounds per 1000 feet
	Bare	Single cotton covered	Double cotton covered	Circular mils	Sq. inches	Cold (25° C. =77° F.)	Hot (65° C. =149° F.)	
18	40	45	48	1620	0.00128	6.51	7.51	4.92
19	36	39	43	1290	0.00101	8.21	9.48	3.90
20	32	36	40	1020	0.000802	10.4	11.9	3.09
21	28.5	32.5	36.5	810	0.000636	13.1	15.1	2.45
22	25.3	29.0	33.0	642	0.000505	16.5	19.0	1.94
23	22.6	26.6	30.6	509	0.000400	20.8	24.0	1.54
24	20.1	24.1	28.1	404	0.000317	26.2	30.2	1.22
25	17.9	21.9	25.9	320	0.000252	33.0	38.1	0.970
26	15.9	19.9	23.9	254	0.000200	41.6	48.0	0.769
27	14.2	18.2	22.2	202	0.000158	52.5	60.6	0.610
28	12.6	16.6	20.6	160	0.000126	66.2	76.4	0.484
29	11.3	15.3	19.3	127	0.0000995	83.4	96.3	0.384
30	10.0	14.0	18.0	101	0.0000789	105	121	0.304
31	8.9	12.9	16.9	79.7	0.0000626	133	153	0.241
32	8.0	11.9	15.9	63.2	0.0000496	167	193	0.191
33	7.1	11.1	15.1	50.1	0.0000394	211	243	0.152
34	6.3	10.3	14.3	39.8	0.0000312	266	307	0.120
35	5.6	9.6	13.6	31.5	0.0000248	335	387	0.0954
36	5.0	8.5	12.0	25.0	0.0000196	423	488	0.0757
37	4.5	....	....	19.8	0.0000156	533	616	0.0600
38	4.0	....	....	15.7	0.0000123	673	776	0.0476
39	3.5	....	....	12.5	0.0000098	848	979	0.0377
40	3.1	....	....	9.9	0.0000078	1070	1230	0.0299

Ohm's Law

$$I = \frac{E}{R} \qquad R = \frac{E}{I}$$

or

$$E = IR$$

I = current in amperes,
E = electromotive force in volts,
R = resistance in ohms.

The proper size of wire in circular mils for any direct current circuit on a two-wire system consisting of copper conductors is given by the formula:

$$\text{c.m.} = \frac{10.8 \times 2\,d \times I}{E}$$

or if the resistance is required,

$$r = \frac{E}{2\,d \times I}$$

where

r = resistance per foot of wire in ohms,
E = volts drop in line,
I = total line current in amperes,
d = distance from source to load in feet,
c.m. = cross-section of conductor in circular mils.

Resistance of Circuits

The resultant of several resistances in **series** equals

$$R = r_1 + r_2 + r_3 + \cdots$$

where r_1, r_2, r_3, etc., are the separate resistances.

The resultant of several resistances in **parallel** or **multiple** is given by the relation:

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \cdots$$

R is the total or combined resistance; and r_1, r_2, r_3, etc., are the separate resistances.

Power and Energy in Direct Current Circuits

The power in watts expended in a resistance is

$$P = EI = I^2R$$

E = electromotive force in volts,
I = current in amperes,
R = resistance in ohms.

The **energy** transformed into heat in a time t seconds is

$$\epsilon = EIt = I^2Rt$$

when the current, I, is constant; or, if the current is variable, energy equals

$$\epsilon = \int_{t_1}^{t_2} i^2R\,dt$$

where i is the instantaneous value of the current, expressed as a function of t.

The **power** in any two-wire direct current circuit is

$$P \text{ (in watts)} = EI$$

where E is the volts between the terminals of the circuit and I is the current in amperes.

MOTORS AND GENERATORS

The **frequency** in cycles per second is given by the relation:

$$f = \frac{\text{R.P.M.}}{60} \times \frac{P}{2}$$

R.P.M. = speed in revolutions per minute,
P = number of poles.

Equations of Direct Current Motor

The **armature current** of a motor, during **starting,** is

$$I_a = \frac{E - e}{R_a + R_x}$$

in which

E = impressed voltage,
e = counter-electromotive force,
R_a = armature resistance in ohms,
R_x = resistance of grid or rheostat in series with armature.

At **full speed,**

$$I_a = \frac{E - e}{R_a}$$

$$e = K\phi f$$

$$E = I_a R_a + e = I_a R_a + K\phi f$$

$$I_a = \frac{E - K\phi f}{R_a}$$

$$f = \frac{E - I_a R_a}{K\phi}$$

f = frequency in cycles per second,*
ϕ = total field flux in magnetic lines, cutting armature conductors,
K = constant for any given machine. Its value is $\frac{4\,t}{10^8}$, where t is the number of armature turns in series.

* Frequency, in the case of a direct current machine, refers to the frequency of alternation in the armature windings, not, of course, in the external circuit.

Equations of Direct Current Generator

$$E = e - I_a R_a$$

e = generated voltage,
E = terminal voltage,
I_a = armature current in amperes,
R_a = armature resistance in ohms.

$$I_a = \frac{E}{R}$$

R = resistance of load in ohms.

$$E = RI_a$$

$$e = E + I_a R_a = I_a (R + R_a)$$

Torque

The torque of a dynamo in foot-pounds equals

$$T = KI\phi$$

where

ϕ = total field flux in magnetic lines, cutting armature conductors,
I = armature current in amperes,
K = constant term for any given dynamo. Its value is $K = \frac{2.348}{10^9} tP$, t being the number of armature turns in series, and P the total number of poles.

The torque of a motor in terms of the horsepower is

$$T = \frac{33{,}000 \text{ H.P.}}{2\pi n}$$

or solving for horsepower,

$$\text{H.P.} = \frac{2\pi Tn}{33{,}000} = \frac{2\pi RFn}{33{,}000}$$

n = number of revolutions per minute,
T = torque in foot-pounds,
R = radius of pulley in feet,
F = turning force in pounds.

Induced Voltage

$$e = -\frac{N}{10^8}\frac{d\phi}{dt} \text{ volts}$$

N = number of turns.

If the turns cut across a uniform field, at right angles to the lines of force, then $\frac{d\phi}{dt}$ equals the number of lines cut per second. Otherwise, $\frac{d\phi}{dt}$ is the first derivative of ϕ in respect to t, ϕ being expressed as a function of t.

The **effective voltage** induced in the windings of a **generator, motor,** or **transformer,** etc., is given by the relation:

$$E = \frac{\sqrt{2}\,\pi f n\phi}{10^8} = \frac{4.44\,fn\phi}{10^8} \text{ volts}$$

This formula is generally quite accurate, being derived on the assumption of uniform flux distribution.

f = frequency in cycles per second,
ϕ = total number of lines of magnetic force,
n = effective number of turns. If all the turns are grouped in one coil, then n equals the total number of turns. Otherwise, if the winding is distributed over k electrical degrees (as in the armature of a motor or generator), then the effective number of turns is $n = N\dfrac{\sin\left(\dfrac{k}{2}\right)}{\dfrac{k}{2}}$, N being the total number of turns.

The **average induced voltage** of a dynamo is

$$E = \frac{4fn\phi}{10^8} \text{ volts}$$

where n is the number of armature turns in series.

Inductance

Inductance, L, is the number of interlinkages of flux with turns, per unit current,

$$L \text{ (henrys)} = \frac{N\phi}{10^8 I}$$

in which

N = number of turns,
I = current in amperes,
ϕ = number of lines of magnetic force interlinking with the turns.

The **theoretical unit** of inductance is the centimeter.

The **practical unit** of inductance is the henry, which equals 10^9 centimeters.

The **counter-electromotive force** in an inductive circuit is

$$e = -L\frac{di}{dt}$$

provided the inductance, L, is constant.

The **total voltage** consumed by an inductive circuit

$$E = ir + L\frac{di}{dt}$$

the inductance, L, being constant.

r is the resistance of the circuit in ohms, and $\frac{di}{dt}$ is the first derivative of i with respect to t, the current i being expressed as a function of t.

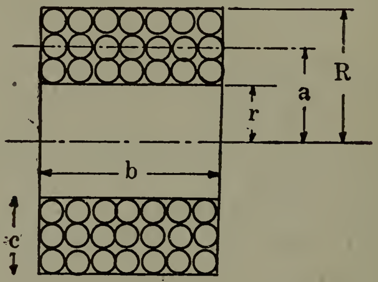
The **inductance** in henrys of an **air-core circular coil** is

$$L = \frac{0.366\left(\frac{l}{1000}\right)^2}{b + c + R} \times F' F''$$

$$F' = \frac{10\,b + 12\,c + 2\,R}{10\,b + 10\,c + 1.4\,R}$$

$$F'' = 0.5 \log_{10}\left(100 + \frac{14\,R}{2\,b + 3\,c}\right)$$

l = length of conductor in feet.



All other dimensions are in inches and as indicated in the diagram.

The **inductance,** L, of a **concentric cable** in henrys per 1000 feet is

$$L = \frac{3.048}{10^5} \times$$

$$\left\{\frac{1}{2} + 4.6 \log_{10}\frac{R}{r} + \frac{4.6\,R_0^4}{(R_0^2 - R^2)^2}\log_{10}\frac{R_0}{R} - \frac{1}{2}\frac{3\,R_0^2 - R^2}{(R_0^2 - R^2)}\right\}$$

where

r = radius of inner metallic conductor,

R = distance from center of cable to the inner surface of the outer metallic conductor,

R_0 = distance from center of cable to the outer surface of the outer metallic conductor.

The values of r, R, and R_0 must be expressed in the same units.

The **total inductance,** L, of a **two-wire transmission circuit** in henrys per 1000 feet is

$$L = \frac{3.048}{10^5}\left\{9.2\,\mu \log_{10}\frac{D-r}{r} + \mu_1\right\}$$

where

μ_1 = permeability of the metal conductor; for copper, $\mu_1 = 1$,

μ = permeability of medium separating wires; for air, $\mu = 1$,

D = distance between the two lines, measured from center to center,

r = radius of conductor, in same unit as D.

Capacity

The **unit of capacity** is the **farad.** Since the farad is very large, the **microfarad,** which is one-millionth of a farad, is used as the practical unit. The **theoretical unit** of capacity is the centimeter, 9×10^{11} centimeters being equal to 1 farad.

The **charge** of a condenser, Q, is measured in ampere-seconds or coulombs, and may be calculated by the formula:

$$Q = CE$$

from which

$$C = \frac{Q}{E}$$

and

$$E = \frac{Q}{C}$$

where

C = capacity in farads,

E = potential across the terminals of the condenser in volts.

The **capacity** of a **plate condenser** is

$$C = \frac{2248\,KA}{d \times 10^{10}} \text{ microfarads}$$

where

A = total area in square inches of **all** the dielectric sheets separating the condenser plates,

d = average thickness in inches of one sheet of the dielectric,

K = inductivity of the dielectric, average values of which are given in the following table for different materials.

Materials	Inductivity K
Air (at standard pressure).	1.00
Manilla paper.	1.50
Paraffin, solid.	2.00
Ebonite.	2.50
India rubber.	2.50
Shellac.	3.00
Oil.	3.00
Glass.	3.10
Mica.	6.00

Condensers in Parallel. When two or more condensers are connected in parallel, the resultant capacity, C, equals the sum of the separate capacities, thus

$$C = C_1 + C_2 + C_3 + \ldots$$

Condensers in Series. When two or more condensers of capacities C_1, C_2, C_3, etc., are connected in series, the resultant capacity is given by the formula:

$$C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots}$$

The **capacity,** C, of a **concentric cable** per 1000 feet in microfarads is

$$C = \frac{7.37}{1000 \log_{10} \frac{\rho_0}{\rho}}$$

in which

ρ = radius of inner metallic conductor,

ρ_0 = distance from center of cable to the inner surface of the outer metallic conductor, in the same unit as ρ.

The **capacity,** C, of a **two-wire transmission line** per 1000 feet in microfarads is given approximately by the formula:

$$C = \frac{3.68}{1000 \log_{10} \frac{D - r}{r}}$$

if the lines are not close to the ground.

D = distance between the two wires of the transmission line, measured from center to center,

r = radius of conductor, in same unit as D.

The **differential equations** of a **condenser** are

$$dq = i\,dt$$

$$q = \text{charge} = \int i\,dt$$

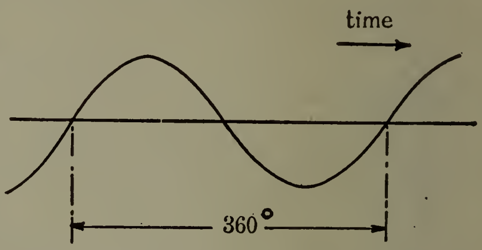
$$dq = c\,de$$

$$i = c\frac{de}{dt}$$

Alternating Current Circuits

The shape of the voltage or current wave produced by an alternator is, in general, nearly that of a **sine curve.** Alternating current calculations are, therefore, usually worked out on this assumption.

The number of cycles or complete waves per second is the **frequency** of the current, and the time required for the current to complete one cycle is a **period.**



The **average value** of the current or voltage is the average of all the ordinates of the curve of one half-wave. The **effective value** of an alternating current or voltage is the square root of the sum of the squares of the instantaneous values of a half-wave.

If E is the maximum voltage of a half-cycle of a sine wave,

$$\textbf{average voltage} = \frac{2}{\pi} E = 0.636\, E$$

$$\textbf{effective voltage} = \frac{1}{\sqrt{2}} E = 0.707\, E$$

Similarly, if the maximum current is I,

$$\textbf{average current} = \frac{2}{\pi} I = 0.636\, I$$

$$\textbf{effective current} = \frac{1}{\sqrt{2}} I = 0.707\, I$$

When the voltage reaches a definite value in the cycle sooner than the current reaches its corresponding value, the voltage and current are **out of phase** with each other; the voltage is said to be **leading,** and the current to be **lagging.** Phase difference is always expressed in degrees; a complete cycle equals 360 degrees.

Alternating Voltage and Current

$$I = \frac{E}{Z} \qquad Z = \frac{E}{I}$$

or

$$E = IZ$$

I = current in amperes,
E = electromotive force in volts,
Z = impedance in ohms.

Impedance and Reactance

r = resistance in ohms
x = reactance in ohms
z = impedance in ohms

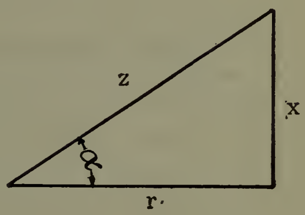
The relation between resistance, reactance, and impedance is the same as that between the three sides of a right triangle.

$$r = z \cos \alpha$$

$$x = z \sin \alpha$$

$$\alpha = \tan^{-1} \frac{x}{r}$$

$$z = \sqrt{r^2 + x^2}$$



Inductive Circuits

The **inductive reactance** in ohms is

$$x_L = 2\pi f L$$

where f = frequency in cycles per second,
L = inductance in henrys.

The **impedance** in ohms is

$$z = \sqrt{r^2 + x_L^2} = \sqrt{r^2 + 4\pi^2 f^2 L^2}$$

Circuits having Capacity

The **capacity reactance** in ohms is

$$x_C = -\frac{1}{2\pi fC}$$

where f = frequency in cycles per second,
C = capacity in farads.

The **impedance** in ohms is

$$z = \sqrt{r^2 + x_C^2} = \sqrt{r^2 + \frac{1}{4\pi^2 f^2 C^2}}$$

Circuits having Inductance and Capacity

The **reactance** in ohms is

$$x = x_L + x_C = 2\pi fL - \frac{1}{2\pi fC}$$

The **impedance** in ohms equals

$$z = \sqrt{r^2 + (x_L + x_C)^2}$$

Vector Representation of Sine Waves

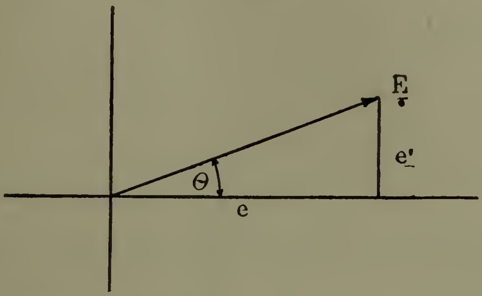
A sine wave of voltage or current may be represented by a vector, the magnitude or length of which is equal to the effective value of the sine wave. It is sometimes more convenient to let the length of the vector equal the maximum value of the sine wave. The vector is generally denoted by a capital letter, with a dot directly beneath it; it is expressed in terms of its rectangular components, which determine the magnitude of the vector and its direction relative to the coördinate axes. Thus, the vector $\underset{\cdot}{E}$ is written

$$\underset{\cdot}{E} = e + je'$$

in which $$j = \sqrt{-1}$$

where e denotes the horizontal or real component of the

vector, and e' the vertical or imaginary component. The imaginary unit, j, in the above equation, merely denotes the direction of measurement of e'.



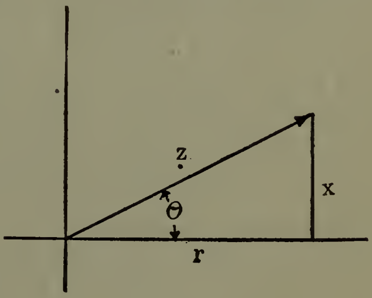
The magnitude of $\underset{\cdot}{E}$ is

$$E = \sqrt{e^2 + e'^2}$$

and the angle θ which the vector $\underset{\cdot}{E}$ makes with the horizontal axis is

$$\theta = \tan^{-1}\frac{e'}{e}$$

The angle in degrees between two vectors is the **phase difference** between the two sine waves which the vectors represent.



In vector notation, the **impedance** is

$$\underset{\cdot}{Z} = r + jx$$

and its magnitude is

$$Z = \sqrt{r^2 + x^2}$$

The **admittance** is

$$\underset{\cdot}{Y} = \frac{1}{\underset{\cdot}{Z}} = \frac{1}{r + jx} = \frac{r}{Z^2} - j\frac{x}{Z^2} = g + jb$$

where $g = \dfrac{r}{Z^2}$ = **conductance,**

$b = -\dfrac{x}{Z^2}$ = **susceptance.**

The **current** equals

$$\dot{I} = \frac{\dot{E}}{\dot{Z}} = \dot{E}\dot{Y} = (e + je')\left(\frac{r}{Z^2} - j\frac{x}{Z^2}\right) = i + ji'$$

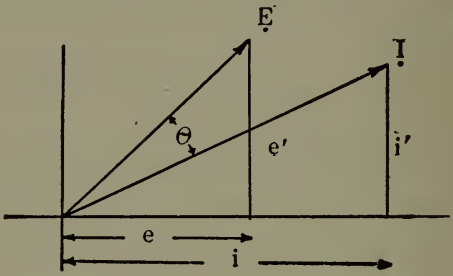
and the **voltage** is

$$\dot{E} = \dot{I}\dot{Z} = (i + ji')(r + jx) = e + je'$$

Power in Alternating Current Circuits

If the effective voltage and current are represented by the vectors

$$\dot{E} = e + je'$$
$$\dot{I} = i + ji'$$



the **real power** is

$$W = ei + e'i' = EI\cos\theta$$

the **wattless power** is

$$W_i = e'i - ei' = EI\sin\theta$$

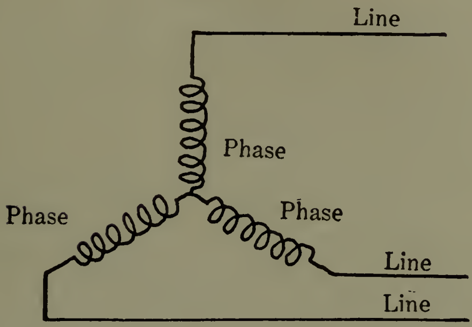
the **volt-amperes** equals EI.

The **power-factor** is the cosine of the angle between the voltage and current vectors,

$$\text{power-factor} = \cos\theta = \frac{ei + e'i'}{EI}$$

Balanced Three-phase Circuits

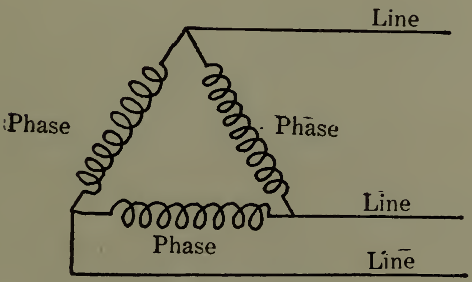
E = volts between lines
e = volts per phase
I = current in each line
i = current in each phase



Y connection

For Y-connections,

$$E = e\sqrt{3}; \quad e = \frac{E}{\sqrt{3}}; \quad \text{and} \quad I = i$$



Δ- connection

For Δ-connections,

$$E = e; \quad I = i\sqrt{3}; \quad \text{and} \quad i = \frac{I}{\sqrt{3}}$$

In either case, for non-inductive load, the power in watts is

$$W = \sqrt{3}\,EI$$

If the load is inductive, then the power is

$$W = \sqrt{3}\,EI\cos\theta$$

where $\cos\theta$ is the power-factor of the phase.

MAGNETISM

Equations of Magnetic Circuits

F = attractive or repellent force in dynes,
mmf = magnetomotive force in ampere turns,
N = number of turns,
I = current in amperes,
β = density in magnetic lines per square centimeter,
ϕ = total number of lines of flux,
A = cross-section of magnetic path in square centimeters,
μ = permeability,
H = intensity of field,
l = length of magnetic circuit in centimeters,
ρ = reluctance,
m = pole strength,
r = distance between poles.

$$\phi = \frac{0.4\,\pi NI}{\rho}$$

$$\rho = \frac{l}{\mu A}$$

$$\phi = \frac{0.4\,\pi NI\mu A}{l}$$

$$\beta = \frac{\phi}{A}$$

$$\beta = \frac{0.4\,\pi NI\mu}{l}$$

$$mmf = 0.4\,\pi NI$$

$$\mu = \frac{\beta}{H}$$

Magnets and Magnetic Fields

$$F = mH$$

$$F = \frac{mm'}{\mu r^2}$$

$$\phi = 4\pi m$$

The **attractive force** in pounds exerted by a **two pole magnet** is $P = \frac{SB^2}{72,134,000}$, where S is the total area of both pole faces in square inches, and B is the density in magnetic lines per square inch.

The **ampere-turns** required to maintain a flux density of B lines per square inch in an **air gap** is $IN = 0.313\ Bl$, in which l is the length of the gap in inches.

Hysteresis Loss

The power in watts lost in hysteresis is

$$W = k\,\frac{fVB^{1.6}}{10^7}$$

f = frequency in cycles per second,
V = volume of iron in cubic inches,
B = magnetic density in lines per square inch,
k = empirical constant, values of which are given in the following table.

Character of iron	Value of k
Silicon steel	0.0006 to 0.00075
Annealed sheet iron	0.0008 to 0.0011
Cast steel	0.010 to 0.012
Cast iron	0.013 to 0.017

Eddy Current Loss

The power in watts lost due to eddy currents in iron or steel laminations is approximately

$$W = \frac{0.00135}{10^7} f^2 l^2 B^2 V$$

f = frequency in cycles per second,
l = average thickness of lamination in inches,
B = magnetic density in lines per square inch,
V = volume of iron in cubic inches.

This formula holds for ordinary temperatures, and if the thickness of the lamination is not greater than 0.025 inch. In silicon steel, the eddy current loss is approximately $\frac{1}{3}$ of that given above.

STANDARD SATURATION CURVES

B = density in lines per square inch
AT/in. = ampere-turns per inch

Values of ampere-turns per inch for densities not included in the following tables may be determined approximately by interpolation. Thus, the AT/in. for silicon steel for B/sq.in. = 65,500 is

$$AT/\text{in.} = 4.5 + \frac{5500}{10{,}000}(6.4 - 4.5) = 5.5 \text{ (approx.)}$$

Silicon Steel		Annealed Sheet Iron	
Saturation curve		Saturation curve	
B	AT/in.	B	AT/in.
30,000	2.1	30,000	4
40,000	2.7	40,000	4.4
50,000	3.4	50,000	5
60,000	4.5	60,000	9
70,000	6.4	70,000	12
80,000	10	80,000	20
90,000	23	90,000	33
100,000	35	100,000	60
110,000	100	....	....
120,000	225	....	....
130,000	520	....	....
135,000	1000	....	....
140,000	2200	....	....
145,000	3770	....	....
150,000	5330	....	....
155,000	6900	....	....

Cast Steel		Cast Iron	
Saturation curve		Saturation curve	
B	AT/in.	B	AT/in.
50,000	11	5,000	8
60,000	15	10,000	12
70,000	20	15,000	17
80,000	29.5	20,000	23
90,000	50	25,000	30
100,000	105	30,000	43
105,000	165	35,000	60
....	....	40,000	85
....	....	45,000	110
....	....	50,000	145
....	....	55,000	190

MEASUREMENT

English Weights and Measures

Length

1000 mils = 1 inch
12 inches = 1 foot
3 feet = 1 yard
5280 feet = 1 mile

4 inches = 1 hand
9 inches = 1 span
$2\frac{1}{2}$ feet = 1 pace
$16\frac{1}{2}$ feet or $5\frac{1}{2}$ yards = 1 rod
1 knot or nautical mile = 6080.26 feet
= $\frac{1}{3}$ league

7.92 inches = 1 link
25 links = 1 rod
100 links or 66 feet or 4 rods = 1 chain
10 chains = 1 furlong
8 furlongs = 1 mile

Surface

144 square inches = 1 square foot
9 square feet = 1 square yard
$30\frac{1}{4}$ square yards = 1 square rod
160 square rods = 1 acre
640 acres = 1 square mile

625 square links = 1 square rod
16 square rods = 1 square chain
10 square chains = 1 acre
640 acres = 1 square mile
36 square miles = 1 township

Volume

1728 cubic inches = 1 cubic foot
27 cubic feet = 1 cubic yard
128 cubic feet = 1 cord
$24\frac{3}{4}$ cubic feet = 1 perch

Troy Weight

24 grains (gr.)	= 1 pennyweight (dwt.)
20 pennyweights	= 1 ounce (oz.)
12 ounces	= 1 pound (lb.)

Avoirdupois Weight

16 drams (dr.)	= 1 ounce (oz.)
16 ounces	= 1 pound (lb.)
25 pounds	= 1 quarter (qr.)
4 quarters	= 1 hundred weight (cwt.)

20 hundred weight (2000 pounds)
= 1 ton (T.)

Apothecaries' Weight

20 grains (gr.)	= 1 scruple (sc. or ℈)
3 scruples	= 1 dram (dr. or ʒ)
8 drams	= 1 ounce (oz. or ℥)
12 ounces	= 1 pound (℔)

Dry Measure

2 pints (pt.)	= 1 quart (qt.)
8 quarts	= 1 peck (pk.)
4 pecks	= 1 bushel (bu.)
36 bushels	= 1 chaldron (ch.)

Liquid Measure

4 gills (gi.)	= 1 pint (pt.)
2 pints	= 1 quart (qt.)
4 quarts	= 1 gallon (gal.)
31½ gallons	= 1 barrel (bar.)
63 gallons	= 1 hogshead (hhd.)

Apothecaries' Fluid Measure

60 minims	= 1 fluid-drachm
8 fluid-drachms	= 1 fluid-ounce
16 fluid-ounces	= 1 pint
8 pints	= 1 gallon

Circular Measure

60 seconds (″)	= 1 minute (′)
60 minutes	= 1 degree (°)
30 degrees	= 1 sign (s)

12 signs, or 360 degrees = 1 circle (cir.)

English and Metric Conversion Factors

Length

1 millimeter	= 39.37 mils
	= 0.03937 inch
1 centimeter	= 0.3937 inch
	= 0.0328 foot
1 inch	= 2.54 centimeters
	= 0.083 foot
1 foot	= 30.48 centimeters
	= 0.305 meter
1 yard	= 91.44 centimeters
	= 0.914 meter
1 meter	= 39.37 inches
	= 3.28 feet
	= 1.094 yards
1 kilometer	= 3280.8 feet
	= 1093.6 yards
	= 0.6214 mile
1 mile	= 5280 feet
	= 1609.3 meters
	= 1.609 kilometer

Surface

1 circular mil	= 0.7854 square mil
	= 0.0005067 square millimeter
1 square mil	= 1.273 circular mils
	= 0.000645 square millimeter
	= 0.000001 square inch
1 sq. millimeter	= 1973 circular mils
	= 1550 sq. mils
	= 0.00155 sq. inch
1 sq. centimeter	= 197,300 circular mils
	= 0.155 sq. inch
1 sq. inch	= 1,273,240 circular mils
	= 6.4516 sq. centimeters
1 sq. foot	= 929.03 sq. centimeters
	= 144 sq. inches

1 sq. yard	=	1296 sq. inches
	=	9 sq. feet
	=	0.00836 are
	=	0.000207 acre
1 sq. meter	=	1550 sq. inches
	=	10.76 sq. feet
	=	1.196 sq. yards
1 are	=	1076 sq. feet
	=	100 sq. meters
1 acre	=	43,560 sq. feet
	=	4840 sq. yards
	=	4047 sq. meters
	=	0.4047 hectare
	=	0.001562 sq. mile
1 hectare	=	107,600 sq. feet
	=	100 ares
	=	2.471 acres
1 sq. kilometer	=	10,764,111 sq. feet
	=	247 acres
	=	0.3861 sq. mile
1 sq. mile	=	27,878,400 sq. feet
	=	640 acres
	=	2.59 sq. kilometers

Volume

1 cu. centimeter	=	0.061 cu. inch
	=	0.0021 pint (liquid)
	=	0.0018 pint (dry)
1 cu. inch	=	16.39 cu. centimeters
	=	0.0173 quart (liquid)
	=	0.01488 quart (dry)
	=	0.0164 liter or cu. decimeter
	=	0.004329 gallon
	=	0.0005787 cu. foot
1 quart (liquid)	=	2 pints (liquid)
	=	946.36 cu. centimeters
	=	57.75 cu. inches
	=	0.94636 liter or cu. decimeter
1 quart (dry)	=	2 pints (dry)

1 quart (dry)	= 1101 cu. centimeters
	= 67.20 cu. inches
	= 0.03889 cu. foot
1 liter	= 1000 cu. centimeters
	= 61.023 cu. inches
	= 1.0567 quarts (dry)
	= 0.2642 gallon
1 gallon	= 3785 cu. centimeters
	= 231 cu. inches
	= 3.785 liters
	= 0.1337 cu. foot

Note. Pints, quarts, and gallons in this table refer to U. S. measures.

Weight

1 milligram	= 0.01543 grain
	= 0.001 gram
1 grain	= 64.80 milligrams
	= 0.002286 ounce (av.)
1 gram	= 15.43 grains
	= 0.03527 ounce (av.)
	= 0.002205 pound (av.)
1 ounce (av.)	= 437.5 grains
	= 28.35 grams
	= 0.0625 pound (av.)
1 pound (av.)	= 7000 grains
	= 453.6 grams
	= 16 ounces
	= 0.4536 kilogram
1 kilogram	= 35.27 ounces
	= 2.205 pounds
1 ton (short)	= 2000 pounds (av.)
	= 907.2 kilograms
	= 0.8928 ton (long)
	= 0.9072 ton (metric)
1 ton (metric)	= 2205 pounds
	= 1000 kilograms
	= 1.102 ton (short)
	= 0.9842 ton (long)

1 ton (long)	= 2240 pounds
	= 1.12 ton (short)
	= 1.016 ton (metric)

Force

1 dyne	= 0.01574 grain
	= 0.00102 gram
	= 0.00007233 poundal
	= 0.000002248 pound (av.)
1 gram	= 980.6 dynes
	= 0.07093 poundal
1 poundal	= 13,825 dynes
	= 0.03108 pound
	= 0.01410 kilogram
1 pound	= 444,800 dynes
	= 32.17 poundals
1 kilogram	= 980600 dynes
	= 70.93 poundals

Storage of Water

1 acre-foot	= 325,800 gallons
	= 43,560 cu. feet
	= 1613 cu. yards
	= 1233 cu. meters
1 gallon	= 0.000003069 acre-foot
1 cu. foot	= 0.00002298 acre-foot
1 cu. yard	= 0.00062 acre-foot

Temperature

1 degree Centigrade	$= \frac{9}{5}$ (= 1.8) degree Fahrenheit
1 degree Fahrenheit	$= \frac{5}{9}$ (= 0.556) degree Centigrade
temperature Fahr.	$= t_f = \frac{9}{5} t_c + 32$
temperature Cent.	$= t_c = \frac{5}{9} (t_f - 32)$

Heat, Electric, and Mechanical Equivalents

Energy

1 erg	= 1 dyne-cm.
	= 0.0000001 joule
	= 0.00000007376 foot-pound
1 gram-centimeter	= 980.6 ergs
	= 0.00009806 joule
	= 0.00007233 foot-pound
1 joule	= 10,000,000 ergs
	= 0.7376 foot-pound
	= 0.2389 gram-calorie
	= 0.102 kilogram-meter
	= 0.0009480 B.t.u.
	= 0.0002778 watt-hour
1 foot-pound	= 13,560,000 ergs
	= 1.356 joules
	= 0.3239 gram-calorie
	= 0.1383 kilogram-meter
	= 0.001285 B.t.u.
	= 0.0003766 watt-hour
	= 0.0000005051 horsepower-hour
1 kilogram-meter	= 9.806 joules
	= 7.233 foot-pounds
	= 0.009296 B.t.u.
	= 0.002724 watt-hour
1 B.t.u.	= 1055 joules
	= 778.1 foot-pounds
	= 252 gram-calories
	= 107.6 kilogram-meters
	= 0.2930 watt-hour
	= 0.0003930 horsepower-hour
1 watt-hour	= 3600 joules
	= 2655.4 foot-pounds
	= 860 gram-calories
	= 3.413 B.t.u.
	= 0.001341 horsepower-hour
1 kilogram-calorie	= 4186 joules
	= 3088 foot-pounds

1 kilogram-calorie	=	426.9 kilogram-meters
	=	1.163 watt-hours
1 horsepower-hour	=	2,684,000 joules
	=	1,980,000 foot-pounds
	=	745.6 watt-hours

Power

1 erg per second	=	1 dyne-centimeter per second
	=	0.0000001 watt
1 gram-centimeter per second	=	0.00009806 watt
1 foot-pound per minute	=	0.02260 watt
	=	0.00003072 horsepower (metric)
	=	0.00003030 horsepower
1 watt	=	44.26 foot-pounds per minute
	=	6.119 kilogram-meters per minute
1 horsepower	=	33,000 foot-pounds per minute
	=	745.6 watts
	=	550 foot-pounds per second
	=	1.01387 horsepower (metric)
1 horsepower (metric)	=	32,550 foot-pounds per minute
	=	735.5 watts
	=	75 kilogram-meters per second
	=	0.9863 horsepower
1 kilowatt	=	44,256.7 foot-pounds per minute
	=	1.3597 horsepower (metric)
	=	1.341 horsepower

Electric Units

1 abvolt	=	10^{-8} volt
1 abampere	=	10 amperes
1 abohm	=	10^{-9} ohm

Pressure Equivalents

1 atmosphere (standard)	=	29.9212 inches of mercury at 32° F.
	=	760 millimeters of mercury at 32° F.
	=	33.901 feet of water at 39.1° F.
	=	14.6969 pounds per sq. inch
	=	2116.35 pounds per sq. foot

1 inch of mercury at 32° F.	= 0.491187 pound per sq. inch
	= 70.7310 pounds per sq. foot
	= 1.13299 feet of water at 39.1° F.
1 foot of water at 39.1° F.	= 0.8826 inch of mercury at 32° F.
	= 62.425 pounds per sq. foot
	= 0.4335 pound per sq. inch
	= 0.0295 atmosphere
1 pound on the sq. foot	= 0.016018 foot of water at 39.1° F.
1 pound on the sq. inch	= 2.307 feet of water at 39.1° F.

PRESSURE AND VOLUME CORRECTION, ETC.

Reduction of Barometer Readings to 0° C.

$$\text{corrected height } H_0 = H\left\{1 - \frac{(\beta - \alpha)\,t}{(1 + \beta t)}\right\}$$

H = observed height of barometer,

t = observed temperature of barometer in degrees Centigrade,

β = 0.0001818, the coefficient of cubical expansion of mercury,

α = coefficient of linear expansion of the material of the scale (0.0000085 for glass, 0.0000184 for brass).

Reduction of Gaseous Volumes to 0° C., and 1 Atmosphere Pressure

$$\text{corrected volume } v_0 = \left\{\frac{v}{1 + 0.00367\,t}\right\}\frac{p}{760}$$

v = observed volume,

t = observed temperature in degrees Centigrade,

p = pressure in millimeters of mercury.

Determination of Altitudes by the Barometer

For heights not exceeding 2000 feet, relative altitude is given by the approximate formula:

$$X \text{ (in feet)} = 52{,}500 \left\{ 1 + \frac{2\,(T + T_1)}{1000} \right\} \frac{H - H_1}{H + H_1}$$

X = vertical distance between the two stations,
T = Centigrade temperature at lower station,
T_1 = Centigrade temperature at upper station,
H = height of barometer at lower station reduced to 0° C.,
H_1 = height of barometer at upper station reduced to 0° C.

For any altitude,

$$X = 60{,}346 \{ 1 + 0.00256 \cos (2\theta) \} \left\{ 1 + 2\frac{(T + T_1)}{1000} \right\} \log_{10} \frac{H}{H_1}$$

in which θ = latitude in degrees.

PHYSICAL AND CHEMICAL CONSTANTS

ATOMIC WEIGHTS

Element	Symbol	Atomic weight	Element	Symbol	Atomic weight
Aluminium..	Al	27.1	Neodymium..	Nd	144.3
Antimony...	Sb	120.2	Neon.........	Ne	20.2
Argon.......	A	39.88	Nickel........	Ni	58.68
Arsenic......	As	74.96	Niobium......	Nb	93.5
Barium......	Ba	137.37	Nitrogen......	N	14.01
Beryllium...	Be	9.1	Osmium......	Os	190.9
Bismuth.....	Bi	208.0	Oxygen.......	O	16.00
Boron.......	B	11.0	Palladium....	Pd	106.7
Bromine.....	Br	79.92	Phosphorous..	P	31.04
Cadmium....	Cd	112.40	Platinum.....	Pt	195.2
Cæsium.....	Cs	132.81	Potassium....	K	39.10
Calcium.....	Ca	40.07	Praseodymium	Pr	140.6
Carbon......	C	12.00	Radium......	Ra	226.4
Cerium......	Ce	140.25	Rhodium.....	Rh	102.9
Chlorine.....	Cl	35.46	Rubidium....	Rb	85.45
Chromium...	Cr	52.0	Ruthenium...	Ru	101.7
Cobalt.......	Co	58.97	Samarium....	Sa	150.4
Copper......	Cu	63.57	Scandium.....	Sc	44.1
Dysprosium..	Dy	162.5	Selenium.....	Se	79.2
Erbium......	Er	167.7	Silicon........	Si	28.3
Europium....	Eu	152.0	Silver........	Ag	107.88
Fluorine.....	F	19.0	Sodium.......	Na	23.00
Gadolinium..	Gd	157.3	Strontium....	Sr	87.63
Gallium.....	Ga	69.9	Sulphur.......	S	32.07
Germanium..	Ge	72.5	Tantalum.....	Ta	181.5
Gold........	Au	197.2	Tellurium.....	Te	127.5
Helium......	He	3.99	Terbium......	Tb	159.2
Hydrogen....	H	1.008	Thallium.....	Tl	204.0
Indium......	In	114.8	Thorium......	Th	232.4
Iodine.......	I	126.92	Thulium......	Tm	168.5
Iridium......	Ir	193.1	Tin...........	Sn	119.0
Iron.........	Fe	55.84	Titanium.....	Ti	48.1
Krypton.....	Kr	82.9	Tungsten.....	W	184.0
Lanthanum..	La	139.0	Uranium......	U	238.5
Lead........	Pb	207.10	Vanadium....	V	51.06
Lithium.....	Li	6.94	Xenon........	Xe	130.2
Lutecium....	Lu	174.0	Ytterbium....	Yb	172.0
Magnesium..	Mg	24.32	Yttrium......	Y	89.0
Manganese..	Mn	54.93	Zinc.........	Zn	65.37
Mercury.....	Hg	200.6	Zirconium....	Zr	90.6
Molybdenum	Mo	96.0			

WEIGHT AND DENSITY OF VARIOUS SUBSTANCES

Values are for ordinary temperatures unless otherwise stated.

Metals	Weight in pounds		Density relative to water
	per cu. in.	per cu. ft.	
Aluminium	0.096	166.5	2.67
Antimony	0.244	422.0	6.76
Bismuth	0.354	612.0	9.82
Brass (ordinary)	0.308	532.0	8.55
Bronze	0.319	552.0	8.85
Calcium	0.057	98.5	1.58
Copper (pure)	0.322	565.0	8.93
Copper (cast) 39° F	0.314	541.5	8.70
Copper (rolled) 39° F	0.321	554.0	8.88
Gold	0.698	1205.0	19.32
Iron (pure)	0.284	490.0	7.86
Iron (cast) 39° F	0.260	449.0	7.21
Iron (wrought) 39° F	0.281	485.0	7.78
Lead	0.411	709.7	11.38
Magnesium	0.064	109.0	1.75
Mercury 32° F	0.491	848.0	13.60
Nickel	0.318	549.0	8.80
Platinum	0.775	1340.0	21.50
Potassium	0.031	53.9	0.87
Silver	0.379	655.0	10.50
Sodium	0.035	60.5	0.97
Steel (hard) 39° F	0.286	494.0	7.92
Steel (soft) 39° F	0.283	488.0	7.83
Tin	0.264	455.0	7.30
Tungsten	0.624	1080.0	17.30
Zinc	0.253	437.0	7.00

WEIGHT AND DENSITY OF VARIOUS SUBSTANCES (*Continued*)

Liquids	Weight in pounds		Density relative to water
	per cu. in.	per cu. ft.	
Acid, hydrochloric	0.0433	74.8	1.20
Acid, nitric	0.0440	76.0	1.22
Acid, sulphuric	0.0675	116.5	1.84
Alcohol	0.0286	49.5	0.79
Carbon disulphide	0.0455	78.5	1.26
Glycerine	0.0455	78.5	1.26
Naphtha	0.0307	53.0	0.85
Oil, linseed	0.0332	57.4	0.92
Oil, lubricating	0.0328	56.6	0.91
Petroleum	0.0307	53.0	0.85
Turpentine	0.0314	54.1	0.87
Water, pure, at			
32° F. (freezing point)	0.036121	62.417	1.0010
39.1° F. (max. density)	0.036125	62.425	1.0011
62° F. (standard temp.)	0.036085	62.355	1.0000
212° F. (boiling point)	0.034549	59.700	0.9574
Water, Sea, 62° F.	0.037023	63.976	1.0260

Values for gases given below are for 32° F. and a pressure of 1 atmosphere.

Gases	Weight in pounds per cu. ft.	Density relative to air
Acetylene, C_2H_2	0.0725	0.898
Air	0.0807	1.000
Ammonia, NH_3	0.0475	0.589
Carbon monoxide, CO	0.0781	0.967
Carbon dioxide, CO_2	0.1227	1.520
Ethylene, C_2H_4	0.0781	0.967
Hydrochloric acid, HCl	0.1023	1.268
Hydrogen, H_2	0.00562	0.0695
Hydrogen sulphide, H_2S	0.0949	1.175
Methane, CH_4	0.0446	0.553
Nitrous oxide, N_2O	0.1235	1.530
Nitric oxide, NO	0.0831	1.030
Nitrogen, N_2	0.0783	0.970
Oxygen, O_2	0.0892	1.105
Sulphur dioxide, SO_2	0.1786	2.210
Water vapor, H_2O	0.0502	0.622

WEIGHT AND DENSITY OF VARIOUS SUBSTANCES (*Continued*)

Woods	Weight in in pounds per cu. ft.	Density relative to water
Ash	45	0.72
Beech	46	0.73
Cedar	39	0.62
Cork	15	0.24
Elm	38	0.61
Fir	37	0.59
Lignum-litæ	62	1.00
Mahogany	51	0.81
Maple	42	0.68
Oak	47	0.75
Pine, Yellow	38	0.61
Pine, White	28	0.45
Poplar	30	0.48
Spruce	28	0.45
Walnut	36	0.58

Other materials	Weight in pounds per cu. ft.	Density relative to water
Asphaltum	87	1.39
Brick, common	112	1.79
Cement, average	90	1.45
Clay	135	2.15
Coal, anthracite	95	1.50
Coal, bituminous	84	1.35
Concrete, average	135	2.20
Earth, loose	75	1.20
Earth, packed	100	1.60
Glass, average	164	2.60
Glass, flint	188	3.02
Granite	165	2.65
Gravel, average	110	1.75
Ice	56	0.90
Limestone	165	2.65
Marble	170	2.73
Quartz	165	2.65
Sand, average	100	1.60
Slate	175	2.80

MELTING AND BOILING POINTS OF ELEMENTS

Element	Melting point		Boiling point at atmospheric pressure	
	Degrees C.	Degrees F.	Degrees C.	Degrees F.
Aluminium	657	1214	1800	3272
Antimony	630	1166	1440	2624
Argon	−188	−306	−186	−303
Arsenic	(volatilizes)		(sublimes 450	842)
Barium	850	1562	.....	.....
Bismuth	269	516	1420	2590
Boron	2000	3630	(sublimes 3500	6330)
Bromine	−7.3	18.8	63	145.5
Cadmium	321	610	778	1432
Calcium	780	1436	.....	.....
Carbon	4000	7230	.....	.....
Chlorine	−102	−151.5	−33.6	−28.5
Chromium	1489	2712	2200	3992
Cobalt	1490	2714	.....	.....
Copper	1083	1982	2310	4190
Fluorine	−223	−370	−187	−305
Gold	1062	1944	2530	4586
Helium	below −270	below −454	−268.6	−452
Hydrogen	−259	−434	−252.7	−423
Iodine	113	235	184.4	364
Iron	1505	2742	2450	4442
Lead	327	621	1525	2776
Lithium	186	367	1400	2552
Magnesium	633	1172	1120	2052
Manganese	1207	2205	1900	3452
Mercury	−38.8	−37.8	356.7	674
Nickel	1452	2648	2330	4226
Nitrogen	−210.5	−347	−195.7	−320
Osmium	2200	3990	.....	.....
Oxygen	−235	−391	−182.9	−297
Palladium	1549	2820	2540	4600
Phosphorous	44.1	111.5	287	549
Platinum	1710	3110	2450	4440
Potassium	62.5	144.5	758	1397
Selenium	217	423	690	1274
Silicon	1420	2588	3500	6330
Silver	960	1760	1955	3551
Sodium	97.0	206.6	877	1612

MELTING AND BOILING POINTS OF ELEMENTS
(*Continued*)

Element	Melting point		Boiling point at atmospheric pressure	
	Degrees C.	Degrees F.	Degrees C.	Degrees F.
Strontium........	900	1650	.....	.....
Sulphur (rhombic)	115	239	445	833
Tantalum........	2910	5270	.....	.....
Tin..............	232	449.6	2270	4122
Titanium.........	2500	4530	.....	.....
Tungsten.........	3083	5582	3700	6700
Zinc..............	418	784	918	1683
Zirconium........	1300	2372	.....	.....

SPECIFIC HEATS

The values of specific heat, unless otherwise stated, are average values, and hold approximately over ordinary ranges of temperatures.

Solids	Specific heat
Aluminium............................	0.219
Antimony............................	0.051
Bismuth.............................	0.0304
Brass...............................	0.094
Copper..............................	0.095
Gold................................	0.032
Iodine..............................	0.054
Iron (wrought)......................	0.114
Iron (cast).........................	0.130
Lead................................	0.031
Magnesium...........................	0.246
Manganese...........................	0.122
Nickel..............................	0.109
Phosphorous.........................	0.189
Platinum............................	0.033
Silicon.............................	0.183
Silver..............................	0.057
Steel...............................	0.117
Sulphur.............................	0.203
Tin.................................	0.056
Tungsten............................	0.034
Zinc................................	0.096

SPECIFIC HEATS (*Continued*)

Liquids	Specific heat
Alcohol, methyl	0.600
Bismuth (melted)	0.0363
Brine (density 1.2) 32° F.	0.710
Lead (melted)	0.0402
Mercury 68° F.	0.0333
Oil, olive	0.47
Sea-water	0.94
Sulphur (melted)	0.234
Tin (melted)	0.064
Turpentine	0.47
Water 32° F.	1.0083
Water 68° F.	0.9992
Water 212° F.	1.0051

Gases	Specific heat at constant pressure	Specific heat at constant volume
Air	0.2375	0.1685
Ammonia	0.508	0.299
Carbon monoxide	0.2479	0.1758
Carbon dioxide	0.217	0.171
Ethylene	0.404	0.332
Hydrogen	3.409	2.412
Nitrogen	0.2438	0.1727
Oxygen	0.2175	0.1550

Other materials	Specific heat
Charcoal	0.241
Glass, crown	0.16
Glass, flint	0.12
Granite	0.19
Ice	0.504
India rubber	0.40
Marble	0.21
Masonry	0.20
Paraffin wax	0.69
Porcelain	0.255
Quartz	0.18

Note. The specific heat of a material is the number of British Thermal Units necessary to raise the temperature of 1 pound of the material 1° F.

Coefficients of Linear Expansion of Solids

The length of a solid at any temperature is $l_t = l_o(1 + \alpha t)$, l_o being the known length at some given temperature, t the variation of temperature in degrees, and α the coefficient of linear expansion of the material. This formula holds approximately when the temperature interval is not large. The coefficient of **surface expansion** equals 2 α; the coefficient of **cubical expansion** equals 3 α.

COEFFICIENTS OF LINEAR EXPANSION (α)

Metals	For 1° C.	For 1° F.
Aluminium	0.0000222	0.0000123
Aluminium bronze	0.000017	0.0000095
Antimony	0.0000113	0.00000627
Bismuth	0.0000176	0.00000975
Brass	0.0000189	0.0000105
Bronze	0.0000177	0.00000985
Carbon, graphite	0.0000079	0.0000044
Copper	0.0000160	0.00000887
German silver (120° F.)	0.0000184	0.0000102
Gold	0.0000142	0.00000786
Gun metal	0.0000181	0.0000101
Iron (cast)	0.0000100	0.00000556
Iron (wrought)	0.0000117	0.00000648
Lead	0.0000283	0.0000157
Nickel	0.0000125	0.00000695
Platinum	0.00000863	0.00000479
Silver	0.0000194	0.0000108
Solder	0.0000250	0.0000139
Steel	0.0000114	0.00000636
Tin	0.0000209	0.0000116
Type metal (275° F.)	0.0000190	0.0000106
Zinc	0.0000253	0.0000141

COEFFICIENTS OF LINEAR EXPANSION (α)
(*Continued*)

Other materials	For 1° C.	For 1° F.
Brick	0.00000550	0.00000305
Concrete	0.0000143	0.00000795
Ebonite	0.0000770	0.0000428
Glass, soft	0.00000850	0.00000470
Glass, hard	0.00000714	0.00000397
Glass, flint	0.00000812	0.00000451
Granite	0.00000789	0.00000438
Ice	0.0000507	0.0000282
Marble	0.0000040	0.0000022
Masonry (average)	0.0000060	0.0000033
Porcelain	0.0000036	0.0000020
Silica (0° to 212° F.)	0.00000050	0.00000028
Slate	0.0000104	0.00000577
Woods, along grain		
beech, mahogany	0.0000030	0.0000017
oak, pine	0.0000050	0.0000028
Woods, across grain		
beech	0.0000600	0.0000330
mahogany	0.0000400	0.0000220
pine	0.0000340	0.0000190

PROPERTIES OF SATURATED STEAM

Tables condensed with permission from G. A. Goodenough's "Properties of Steam and Ammonia," published by Messrs. John Wiley and Sons.

Absolute pressure in inches of mercury	Temp. Fahr.	Volume of one pound in cu. ft., v''	Heat content in B.t.u.		Latent heat in B.t.u.		Entropy		
			of liquid, i'	of vapor, i''	Total, L or r	Internal, I or ρ	of liquid, n or s'	of vaporization, $\frac{L}{T}$ or $\frac{r}{T}$	of vapor, N or s''
1	79.06	652	47.11	1095.0	1047.9	988.7	0.0915	1 9455	2.0370
2	101.17	338.9	69.16	1105.1	1036.0	974.3	0.1316	1.8474	1.9790
3	115.08	231.4	83.04	1111.4	1028.3	965.2	0.1561	1.7893	1.9454
4	125.44	176.5	93.37	1115.9	1022.5	958.3	0.1739	1.7478	1.9217
5	133.78	143.2	101.68	1119.6	1017.9	952.8	0.1380	1.7154	1.9034
6	140.80	120.7	108.69	1122.6	1013.9	948.1	0.1998	1.6888	1.8886
7	146.88	110.4	114.8	1125.2	1010.5	944.0	0.2098	1.6661	1.8760
8	152.26	92.1	120.2	1127.5	1007.4	940.4	0.2187	1.6464	1.8651
9	157.10	82.5	125.0	1129.6	1004.6	937.1	0.2265	1.6290	1.8556
10	161.50	74.8	129.4	1131.4	1002.1	934.1	0.2336	1.6134	1.8470
11	165.55	68.4	133.4	1133.1	999.7	931.3	0.2401	1.5992	1.8393
12	169.30	63.0	137.2	1134.7	997.5	928.8	0.2461	1.5862	1.8323
13	172.79	58.5	140.7	1136.1	995.5	926.4	0.2516	1.5742	1.8258
14	176.06	54.6	143.9	1137.5	993.6	924.1	0.2568	1.5630	1.8198
15	179.14	51.14	147.0	1138.8	991.7	922.0	0.2617	1.5526	1.8143
16	182.06	48.14	149.9	1140.0	990.0	920.0	0.2662	1.5429	1.8091
17	184.83	45.49	152.7	1141.1	988.3	918.1	0.2705	1.5337	1.8042
18	187.46	43.12	155.4	1142.1	986.7	916.2	0.2746	1.5250	1.7996
19	189.97	40.99	157.9	1143.1	985.2	914.4	0.2785	1.5168	1.7953
20	192.38	39.08	160.3	1144.1	983.8	912.7	0.2822	1.5089	1.7912

PROPERTIES OF SATURATED STEAM (*Continued*)

Absolute pressure in inches of mercury	Temp. Fahr.	Volume of one pound in cu. ft., v''	Heat content in B.t.u.		Latent heat in B.t.u.		Entropy		
			of liquid, i'	of vapor, i''	Total, L or r	Internal, I or ρ	of liquid, n or s'	of vaporization, $\frac{L}{T}$ or $\frac{r}{T}$	of vapor, N or s''
21	194.68	37.34	162.6	1145.0	982.4	911.1	0.2858	1.5015	1.7873
22	196.89	35.75	164.8	1145.9	981.1	909.6	0.2892	1.4944	1.7835
23	199.03	34.29	167.0	1146.7	979.8	908.1	0.2924	1.4876	1.7800
24	201.09	32.95	169.0	1147.5	978.5	906.6	0.2955	1.4810	1.7766
25	203.08	31.71	170.1	1148.3	977.3	905.2	0.2986	1.4747	1.7733
26	205.00	30.57	173.0	1149.1	976.1	903.8	0.3015	1.4687	1.7702
27	206.87	29.51	174.8	1149.8	974.9	902.5	0.3043	1.4629	1.7671
28	208.67	28.53	176.6	1150.5	973.8	901.2	0.3070	1.4572	1.7642
29	210.43	27.61	178.4	1151.2	972.7	900.0	0.3096	1.4518	1.7614
in pounds per sq. inch									
14.7	212.0	26.81	180.0	1151.7	971.7	898.8	0.3120	1.4469	1.7589
15	213.0	26.30	181.0	1152.2	971.2	898.1	0.3135	1.4438	1.7573
16	216.3	24.76	184.3	1153.4	969.1	895.8	0.3184	1.4337	1.7521
17	219.4	23.40	187.5	1154.6	967.1	893.5	0.3230	1.4242	1.7473
18	222.4	22.18	190.5	1155.7	965.2	891.4	0.3274	1.4153	1.7427
19	225.2	21.09	193.3	1156.7	963.4	889.3	0.3316	1.4068	1.7384
20	228.0	20.10	196.0	1157.7	961.7	887.3	0.3356	1.3987	1.7343
22	233.1	18.38	201.2	1159.6	958.4	883.6	0.3430	1.3837	1 7267
24	237.8	16.95	206.0	1161.3	955.3	880.1	0.3499	1.3698	1.7197
26	242.2	15.73	210.4	1162.8	952.4	876.8	0.3563	1.3570	1.7133
28	246.4	14.67	214.6	1164.3	949.7	873.7	0.3622	1.3452	1.7074

PROPERTIES OF SATURATED STEAM (*Continued*)

Absolute pressure in pounds per sq. in.	Temp. Fahr.	Volume of one pound in cu. ft., v''	Heat content in B.t.u.		Latent heat in B.t.u.		Entropy		
			of liquid, i'	of vapor, i''	Total, L or r	Internal, I or ρ	of liquid, n or s'	of vaporization, $\frac{L}{T}$ or $\frac{r}{T}$	of vapor, N or s''
30	250.3	13.76	218.6	1165.7	947.1	870.7	0.3679	1.3340	1.7019
32	254.0	12.95	222.4	1166.9	944.6	867.9	0.3731	1.3236	1.6967
34	257.6	12.24	225.9	1168.1	942.2	865.2	0.3781	1.3137	1.6918
36	260.9	11.60	229 4	1169.2	939.9	862.7	0.3829	1.3044	1.6873
38	264.2	11.03	232.6	1170.3	937.7	860.2	0.3874	1.2956	1.6830
40	267.2	10.51	235.8	1171.3	935.5	857.8	0.3917	1.2871	1.6788
42	270.2	10.04	238.8	1172.2	933.5	855.5	0.3958	1.2791	1.6749
44	273.0	9.61	241.7	1173.2	931.5	853.3	0.3998	1.2714	1.6712
46	275.8	9.22	244.5	1174.0	929.6	851.2	0.4036	1.2640	1.6676
48	278.4	8.86	247.2	1174.8	927.7	849.1	0.4072	1.2570	1.6642
50	281.0	8.53	249.8	1175.6	925.9	847.1	0.4108	1.2501	1.6609
52	283.5	8.22	252.3	1176.4	924.1	845.1	0.4142	1.2436	1.6577
54	285.9	7.93	254.7	1177.1	922.4	843.2	0.4174	1.2373	1.6547
56	288.2	7.67	257.1	1177.8	920.7	841.4	0.4206	1.2311	1.6517
58	290.5	7.42	259.5	1178.5	919.0	839.5	0.4237	1.2252	1.6489
60	292.7	7.18	261.7	1179.1	917.4	837.8	0.4267	1.2195	1.6462
62	294.9	6.97	263.9	1179.7	915.8	836.0	0.4296	1.2139	1.6435
64	296.9	6.76	266.1	1180.3	914.3	834.3	0.4324	1.2085	1.6409
66	299.0	6.57	268.2	1180.9	912.7	832.7	0.4352	1.2032	1.6384
68	301.0	6.39	270.2	1181.5	911.2	831.1	0.4379	1.1981	1.6360

PROPERTIES OF SATURATED STEAM (*Continued*)

Absolute pressure in pounds per sq. in.	Temp. Fahr.	Volume of one pound in cu. ft., v''	Heat content in B.t.u.		Latent heat in B.t.u.		Entropy		
			of liquid, i'	of vapor, i''	Total, L or r	Internal, I or ρ	of liquid, n or s'	of vaporization, $\frac{L}{T}$ or $\frac{r}{T}$	of vapor, N or s''
70	302.9	6.22	272.2	1182.0	909.8	829.5	0.4405	1.1931	1.6336
72	304.8	6.05	274.2	1182.5	908.3	827.9	0.4431	1.1883	1.6313
74	306.7	5.90	276.1	1183.0	906.9	826.4	0.4456	1.1835	1.6291
76	308.5	5.75	278.0	1183.5	905.5	824.9	0.4480	1.1789	1.6269
78	310.3	5.61	279.8	1184.0	904.2	823.4	0.4504	1.1744	1.6248
80	312.0	5.48	281.6	1184.4	902.8	821.9	0.4527	1.1700	1.6227
82	313.7	5.35	283.4	1184.9	901.5	820.5	0.4550	1.1657	1.6207
84	315.4	5.23	285.1	1185.3	900.2	819.1	0.4572	1.1615	1.6187
86	317.1	5.12	286.8	1185.7	898.9	817.7	0.4594	1.1574	1.6168
88	318.7	5.01	288.5	1186.1	897.7	816.3	0.4615	1.1534	1.6149
90	320.3	4.905	290.1	1186.5	896.4	815.0	0.4636	1.1495	1.6131
92	321.8	4.805	291.7	1186.9	895.2	813.7	0.4657	1.1456	1.6113
94	323.3	4.709	293.3	1187.3	894.0	812.4	0.4677	1.1419	1.6096
96	324.8	4.617	294.8	1187.7	892.8	811.1	0.4697	1.1381	1.6079
98	326.3	4.528	296.4	1188.0	891.6	809.8	0.4717	1.1345	1.6062
100	327.8	4.442	297.9	1188.4	890.5	808.6	0.4736	1.1309	1.6045
102	329.2	4.359	299.4	1188.7	889.3	807.4	0.4755	1.1274	1.6028
104	330.7	4.279	300.9	1189.0	888.2	806.1	0.4773	1.1239	1.6012
106	332.0	4.202	302.3	1189.4	887.1	804.9	0.4791	1.1205	1.5996
108	333.4	4.128	303.7	1189.7	885.9	803.8	0.4809	1.1172	1.5981

PROPERTIES OF SATURATED STEAM (*Continued*)

Absolute pressure in pounds per sq. in.	Temp. Fahr.	Volume of one pound in cu. ft., v''	Heat content in B.t.u.		Latent heat in B.t.u.		Entropy		
			of liquid, i'	of vapor, i''	Total, L or r	Internal, I or ρ	of liquid, n or s'	of vaporization, $\frac{L}{T}$ or $\frac{r}{T}$	of vapor, N or s''
110	334.8	4.057	305.1	1190.0	884.8	802.6	0.4827	1.1138	1.5965
112	336.1	3.988	306.5	1190.3	883.7	801.4	0.4844	1.1106	1.5950
114	337.4	3.921	307.9	1190.6	882.7	800.3	0.4861	1.1074	1.5935
116	338.7	3.857	309.2	1190.8	881.6	799.2	0.4878	1.1043	1.5921
118	340.0	3.795	310.6	1191.1	880.6	798.0	0.4895	1.1012	1.5907
120	341.3	3.735	311.9	1191.4	879.5	796.9	0.4911	1.0982	1.5893
122	342.5	3.676	313.2	1191.6	878.5	795.8	0.4927	1.0952	1.5879
124	343.7	3.620	314.4	1191.9	877.5	794.8	0.4943	1.0922	1.5865
126	345.0	3.566	315.7	1192.1	876.4	793.7	0.4958	1.0894	1.5852
128	346.2	3.513	316.9	1192.4	875.4	792.6	0.4974	1.0865	1.5838
130	347.4	3.461	318.2	1192.6	874.4	791.6	0.4989	1.0836	1.5825
132	348.5	3.412	319.4	1192.9	873.5	790.5	0.5004	1.0808	1.5812
134	349.7	3.363	320.6	1193.1	872.5	789.5	0.5019	1.0781	1.5800
136	350.8	3.316	321.8	1193.3	871.5	788.5	0.5033	1.0754	1.5787
138	352.0	3.270	323.0	1193.5	870.5	787.4	0.5048	1.0727	1.5775
140	353.1	3.226	324.2	1193.7	869.6	786.4	0.5062	1.0700	1.5762
142	354.2	3.182	325.3	1193.9	868.6	785.4	0.5076	1.0674	1.5750
144	355.3	3.140	326.5	1194.1	867.7	784.5	0.5090	1.0648	1.5738
146	356.3	3.099	327.6	1194.3	866.8	783.5	0.5104	1.0623	1.5727
148	357.4	3.059	328.7	1194.5	865.8	782.5	0.5117	1.0598	1.5715

PROPERTIES OF SATURATED STEAM (*Continued*)

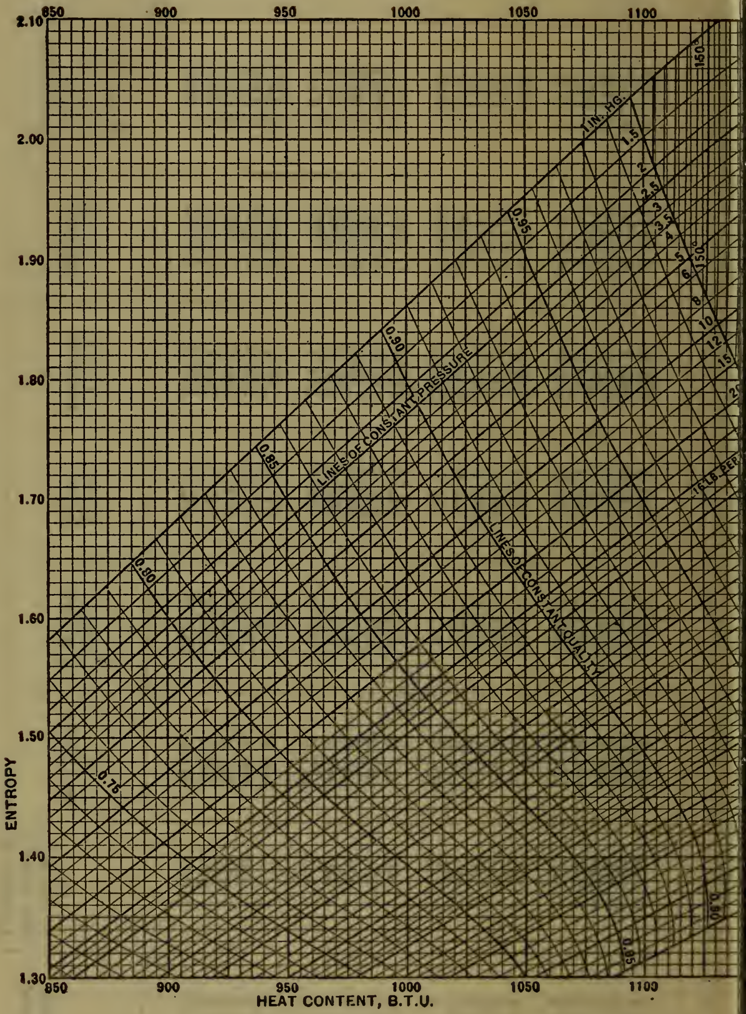
Absolute pressure in pounds per sq. in.	Temp. Fahr.	Volume of one pound in cu. ft., v''	Heat content in B.t.u.		Latent heat in B.t.u.		Entropy		
			of liquid, i'	of vapor, i''	Total, L or r	Internal, I or ρ	of liquid, n or s'	of vaporization, $\frac{L}{T}$ or $\frac{r}{T}$	of vapor, N or s''
150	358.5	3.020	329.8	1194.7	864.9	781.6	0.5131	1.0573	1.5704
152	359.5	2.982	330.9	1194.9	864.0	780.6	0.5144	1.0548	1.5692
154	360.5	2.945	332.0	1195.1	863.1	779.7	0.5157	1.0524	1.5681
156	361.6	2.909	333.1	1195.3	862.3	778.7	0.5170	1.0500	1.5670
158	362.6	2.874	334.1	1195.5	861.4	777.8	0.5183	1.0476	1.5659
160	363.6	2.839	335.2	1195.7	860.5	776.9	0.5196	1.0453	1.5649
162	364.6	2.806	336.2	1195.8	859.6	776.0	0.5209	1.0429	1.5638
164	365.6	2.773	337.3	1196.0	858.7	775.1	0.5221	1.0406	1.5627
166	366.5	2.741	338.3	1196.2	857.9	774.2	0.5233	1.0384	1.5617
168	367.5	2.710	339.3	1196.3	857.0	773.3	0.5245	1.0361	1.5607
170	368.5	2.679	340.3	1196.5	856.2	772.4	0.5258	1.0339	1.5597
172	369.4	2.649	341.3	1196.6	855.3	771.5	0.5270	1.0317	1.5587
174	370.4	2.620	342.3	1196.8	854.5	770.6	0.5281	1.0295	1.5577
176	371.3	2.591	343.3	1196.9	853.6	769.8	0.5293	1.0274	1.5567
178	372.2	2.563	344.3	1197.1	852.8	768.9	0.5305	1.0252	1.5557
180	373.1	2.536	345.2	1197.2	852.0	768.0	0.5316	1.0231	1.5547
182	374.0	2.509	346.2	1197.4	851.2	767.2	0.5328	1.0210	1.5538
184	374.9	2.483	347.1	1197.5	850.4	766.4	0.5339	1.0189	1.5528
186	375.8	2.457	348.1	1197.6	849.5	765.5	0.5350	1.0169	1.5519
188	376.7	2.432	349.0	1197.8	848.7	764.7	0.5361	1.0148	1.5509

PROPERTIES OF SATURATED STEAM (*Continued*)

Absolute pressure in pounds per sq. in.	Temp. Fahr.	Volume of one pound in cu. ft., v''	Heat content in B.t.u.		Latent heat in B.t.u.		Entropy		
			of liquid, i'	of vapor, i''	Total, L or r	Internal, I or ρ	of liquid, n or s'	of vaporization, $\frac{L}{T}$ or $\frac{r}{T}$	of vapor, N or s'
190	377.6	2.408	350.0	1197.9	847.9	763.9	0.5372	1.0128	1.5500
192	378.5	2.383	350.9	1198.0	847.1	763.0	0.5383	1.0108	1.5491
194	379.3	2.360	351.8	1198.1	846.3	762.2	0.5394	1.0089	1.5482
196	380.2	2.337	352.7	1198.2	845.6	761.4	0.5404	1.0069	1.5473
198	381.0	2.314	353.6	1198.4	844.8	760.6	0.5415	1.0049	1.5464
200	381.9	2.292	354.5	1198.5	844.0	759.8	0.5426	1.0030	1.5456
205	383.9	2.238	356.7	1198.7	842.1	757.8	0.5451	0.9983	1.5434
210	386.0	2.186	358.8	1199.0	840.2	755.9	0.5477	0.9936	1.5413
215	388.0	2.137	361.0	1199.2	838.3	754.0	0.5502	0.9890	1.5392
220	390.0	2.090	363.0	1199.5	836.5	752.1	0.5526	0.9846	1.5372
225	391.9	2.045	365.1	1199.7	834.6	750.2	0.5550	0.9802	1.5352
230	393.8	2.002	367.1	1199.9	832.8	748.3	0.5573	0.9760	1.5333
235	395.6	1.961	369.1	1200.1	831.0	746.5	0.5597	0.9717	1.5314
240	397.5	1.921	371.0	1200.3	829.3	744.7	0.5619	0.9676	1.5295
245	399.3	1.883	373.0	1200.5	827.5	742.9	0.5641	0.9635	1.5276
250	401.1	1.846	374.9	1200.6	825.8	741.2	0.5663	0.9595	1.5258
255	402.9	1.811	376.7	1200.8	824.1	739.5	0.5685	0.9556	1.5241
260	404.5	1.777	378.6	1201.0	822.4	737.7	0.5706	0.9517	1.5223
265	406.2	1.745	380.4	1201.1	820.7	736.0	0.5727	0.9479	1.5206
270	407.9	1.713	382.2	1201.2	819.1	734.4	0.5747	0.9442	1.5189

PROPERTIES OF SATURATED STEAM (*Continued*)

Absolute pressure in pounds per sq. in.	Temp. Fahr.	Volume of one pound in cu. ft., v''	Heat content in B.t.u.		Latent heat in B.t.u.		Entropy		
			of vapor, i'	of liquid, i''	Total, L or r	Internal, I or ρ	of liquid, n or s'	of vaporization, $\frac{L}{T}$ or $\frac{r}{T}$	of vapor, N or s''
275	409.6	1.683	383.9	1201.4	817.4	732.7	0.5767	0.9405	1.5172
280	411.2	1.654	385.7	1201.5	815 8	731.1	0.5787	0.9369	1.5156
285	412.8	1.625	387.4	1201.6	814.2	729.5	0.5806	0.9333	1.5139
290	414.4	1.598	389.1	1201.7	812.6	727.9	0.5826	0.9298	1 5123
295	415.9	1.571	390.8	1201.8	811.0	726.3	0.5845	0.9263	1.5108
300	417.5	1.545	392.4	1201.9	809.4	724.7	0.5863	0.9229	1.5092
310	420.5	1.496	395.7	1202.0	806.4	721.6	0.5900	0.9162	1.5062
320	423.4	1.450	398.9	1202.2	803.3	718.5	0.5935	0.9097	1.5032
330	426.3	1.407	402.0	1202.3	800.3	715.6	0.5970	0.9034	1.5004
340	429.1	1.366	405.0	1202.4	797.4	712.6	0.6004	0.8972	1.4976
350	431.9	1.327	408.0	1202.5	794.5	709.7	0.6036	0.8912	1.4949
360	434.6	1.291	410.9	1202.5	791.6	706.9	0.6068	0.8854	1.4922
370	437.2	1.256	413.7	1202.6	788.8	704.1	0.6100	0.8796	1.4896
380	439.8	1.223	416.5	1202.6	786.1	701.4	0.6130	0.8741	1.4871
390	442.3	1.192	419.3	1202.6	783.3	698.7	0.6161	0.8686	1.4847
400	444.8	1.162	422.0	1202.5	780.6	695.9	0.6190	0.8631	1.4821
450	456.5	1.033	434.8	1202.2	767.4	683.1	0.6329	0.8377	1.4706
500	467.2	0.928	446.6	1201.7	755.0	670.9	0.6455	0.8146	1.4601
600	486.5	0.770	468.0	1199.8	731.8	648.5	0.6679	0.7735	1.4414
700	503.4	0.656	487.1	1197.4	710.3	627.9	0.6874	0.7376	1.4250



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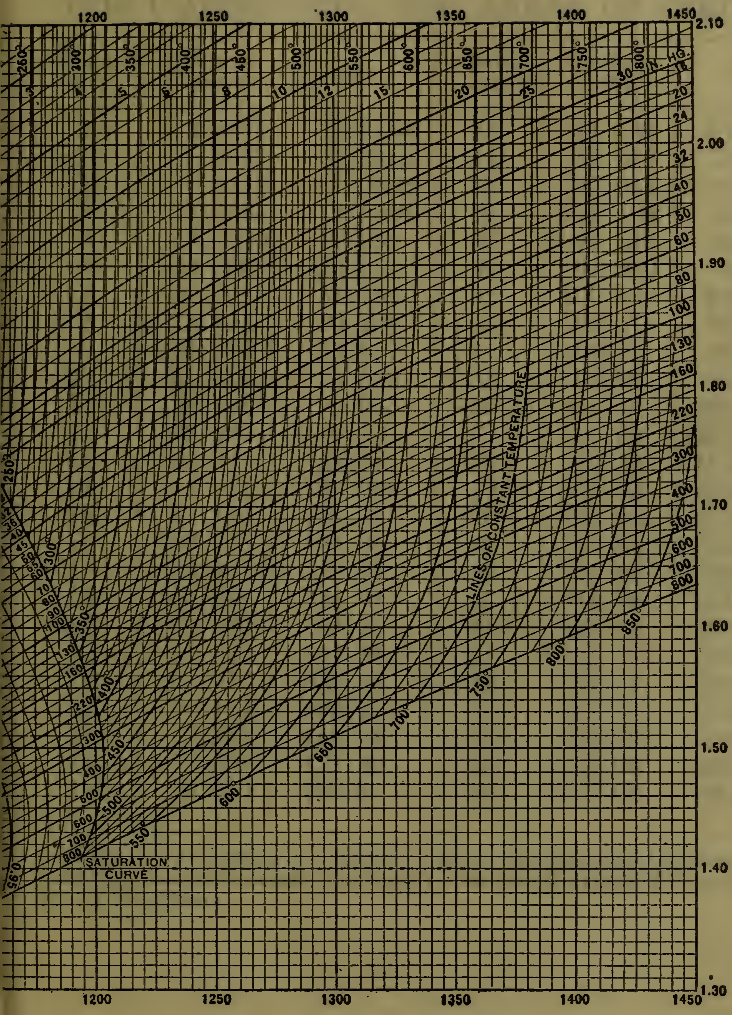

AM CHART

"Properties of Steam and Ammonia," published by
iley & Sons.

TABLES

CIRCUMFERENCES AND AREAS OF CIRCLES

Diameter	Circumference	Area	Diameter	Circumference	Area
1	3.1416	0.7854	26	81.681	530.93
2	6 2832	3.1416	27	84.823	572.56
3	9.4248	7.0686	28	87.965	615.75
4	12.5664	12.5664	29	91.106	660.52
5	15.7080	19.635	30	94.248	706.86
6	18.850	28.274	31	97.389	754.77
7	21.991	38.485	32	100.53	804.25
8	25.133	50.266	33	103.67	855.30
9	28.274	63.617	34	106.81	907.92
10	31.416	78.540	35	109.96	962.11
11	34.558	95.033	36	113.10	1017.88
12	37.699	113.10	37	116.24	1075.21
13	40.841	132.73	38	119.38	1134.11
14	43.982	153.94	39	122.52	1194.59
15	47.124	176.71	40	125.66	1256.64
16	50.265	201.06	41	128.81	1320.25
17	53.407	226.98	42	131.95	1385.44
18	56.549	254.47	43	135.09	1452.20
19	59.690	283.53	44	138.23	1520.53
20	62.832	314.16	45	141.37	1590.43
21	65.973	346.36	46	144.51	1661.90
22	69.115	380.13	47	147.65	1734.94
23	72.257	415.48	48	150.80	1809.56
24	75.398	452.39	49	153.94	1885.74
25	78.540	490.87	50	157.08	1963.50

Note. — The surface of a sphere of given diameter may be found directly from the above table, since it is equal to the area of a circle of twice the diameter of the sphere.

CIRCUMFERENCES AND AREAS OF CIRCLES
(*Continued*)

Diameter	Circumference	Area	Diameter	Circumference	Area
51	160.22	2042.82	76	238.76	4536.46
52	163.36	2123.72	77	241.90	4656.63
53	166.50	2206.18	78	245.04	4778.36
54	169.65	2290.22	79	248.19	4901.67
55	172.79	2375.83	80	251.33	5026.55
56	175.93	2463.01	81	254.47	5153.00
57	179.07	2551.76	82	257.61	5281.02
58	182.21	2642.08	83	260.75	5410.61
59	185.35	2733.97	84	263.89	5541.77
60	188.50	2827.43	85	267.04	5674.50
61	191.64	2922.47	86	270.18	5808.80
62	194.78	3019.07	87	273.32	5944.68
63	197.92	3117.25	88	276.46	6082.12
64	201.06	3216.99	89	279.60	6221.14
65	204.20	3318.31	90	282.74	6361.73
66	207.34	3421.19	91	285.88	6503.88
67	210.49	3525.65	92	289.03	6647.61
68	213.63	3631.68	93	292.17	6792.91
69	216.77	3739.28	94	295.31	6939.78
70	219.91	3848.45	95	298.45	7088.22
71	223.05	3959.19	96	301.59	7238.23
72	226.19	4071.50	97	304.73	7389.81
73	229.34	4185.39	98	307.88	7542.96
74	232.48	4300.84	99	311.02	7697.69
75	235.62	4417.86	100	314.16	7853.98

POWERS, ROOTS, AND RECIPROCALS

Number	Square	Cube	Square root	Cube root	Reciprocal
1	1	1	1.000000	1.000000	1.0000000
2	4	8	1.414214	1.259921	.5000000
3	9	27	1.732051	1.442250	.3333333
4	16	64	2.000000	1.587401	.2500000
5	25	125	2.236068	1.709976	.2000000
6	36	216	2.449490	1.817121	.1666667
7	49	343	2.645751	1.912931	.1428571
8	64	512	2.828427	2.000000	.1250000
9	81	729	3.000000	2.080084	.1111111
10	100	1000	3.162278	2.154435	.1000000
11	121	1331	3.316625	2.223980	.0909091
12	144	1728	3.464102	2.289429	.0833333
13	169	2197	3.605551	2.351335	.0769231
14	196	2744	3.741657	2.410142	.0714286
15	225	3375	3.872983	2.466212	.0666667
16	256	4096	4.000000	2.519842	.0625000
17	289	4913	4.123106	2.571282	.0588235
18	324	5832	4.242641	2.620741	.0555556
19	361	6859	4.358899	2.668402	.0526316
20	400	8000	4.472136	2.714418	.0500000
21	441	9261	4.582576	2.758924	.0476190
22	484	10,648	4.690416	2.802039	.0454545
23	529	12,167	4.795832	2.843867	.0434783
24	576	13,824	4.898980	2.884499	.0416667
25	625	15,625	5.000000	2.924018	.0400000
26	676	17,576	5.099020	2.962496	.0384615
27	729	19,683	5.196152	3.000000	.0370370
28	784	21,952	5.291503	3.036589	.0357143
29	841	24,389	5.385165	3.072317	.0344828
30	900	27,000	5.477226	3.107233	.0333333
31	961	29,791	5.567764	3.141381	.0322581
32	1024	32,768	5.656854	3.174802	.0312500
33	1089	35,937	5.744563	3.207534	.0303030
34	1156	39,304	5.830952	3.239612	.0294118
35	1225	42,875	5.916080	3.271066	.0285714
36	1296	46,656	6.000000	3.301927	.0277778
37	1369	50,653	6.082763	3.332222	.0270270

POWERS, ROOTS, AND RECIPROCALS
(Continued)

Number	Square	Cube	Square root	Cube root	Reciprocal
38	1444	54,872	6.164414	3.361975	.0263158
39	1521	59,319	6.244998	3.391211	.0256410
40	1600	64,000	6.324555	3.419952	.0250000
41	1681	68,921	6.403124	3.448217	.0243902
42	1764	74,088	6.480741	3.476027	.0238095
43	1849	79,507	6.557439	3.503398	.0232558
44	1936	85,184	6.633250	3.530348	.0227273
45	2025	91,125	6.708204	3.556893	.0222222
46	2116	97,336	6.782330	3.583048	.0217391
47	2209	103,823	6.855655	3.608826	.0212766
48	2304	110,592	6.928203	3.634241	.0208333
49	2401	117,649	7.000000	3.659306	.0204082
50	2500	125,000	7.071068	3.684031	.0200000
51	2601	132,651	7.141428	3.708430	.0196078
52	2704	140,608	7.211103	3.732511	.0192308
53	2809	148,877	7.280110	3.756286	.0188679
54	2916	157,464	7.348469	3.779763	.0185185
55	3025	166,375	7.416199	3.802953	.0181818
56	3136	175,616	7.483315	3.825862	.0178571
57	3249	185,193	7.549834	3.848501	.0175439
58	3364	195,112	7.615773	3.870877	.0172414
59	3481	205,379	7.681146	3.892997	.0169492
60	3600	216,000	7.745967	3.914868	.0166667
61	3721	226,981	7.810250	3.936497	.0163934
62	3844	238,328	7.874008	3.957892	.0161290
63	3969	250,047	7.937254	3.979057	.0158730
64	4096	262,144	8.000000	4.000000	.0156250
65	4225	274,625	8.062258	4.020726	.0153846
66	4356	287,496	8.124038	4.041240	.0151515
67	4489	300,763	8.185353	4.061548	.0149254
68	4624	314,432	8.246211	4.081655	.0147059
69	4761	328,509	8.306624	4.101566	.0144928
70	4900	343,000	8.366600	4.121285	.0142857
71	5041	357,911	8.426150	4.140818	.0140845
72	5184	373,248	8.485281	4.160168	.0138889
73	5329	389,017	8.544004	4.179339	.0136986

POWERS, ROOTS, AND RECIPROCALS (*Continued*)

Number	Square	Cube	Square root	Cube root	Reciprocal
74	5476	405,224	8.602325	4.198336	.0135135
75	5625	421,875	8.660254	4.217163	.0133333
76	5776	438,976	8.717798	4.235824	.0131579
77	5929	456,533	8.774964	4.254321	.0129870
78	6084	474,552	8.831761	4.272659	.0128205
79	6241	493,039	8.888194	4.290840	.0126582
80	6400	512,000	8.944272	4.308870	.0125000
81	6561	531,441	9.000000	4.326749	.0123457
82	6724	551,368	9.055385	4.344482	.0121951
83	6889	571,787	9.110434	4.362071	.0120482
84	7056	592,704	9.165151	4.379519	.0119048
85	7225	614,125	9.219545	4.396830	.0117647
86	7396	636,056	9.273619	4.414005	.0116279
87	7569	658,503	9.327379	4.431048	.0114943
88	7744	681,472	9.380832	4.447960	.0113636
89	7921	704,969	9.433981	4.464745	.0112360
90	8100	729,000	9.486833	4.481405	.0111111
91	8281	753,571	9.539392	4.497941	.0109890
92	8464	778,688	9.591663	4.514357	0108696
93	8649	804,357	9.643651	4.530655	.0107527
94	8836	830,584	9.695360	4.546836	.0106383
95	9025	857,375	9.746794	4.562903	.0105263
96	9216	884,736	9.797959	4.578857	.0104167
97	9409	912,673	9.848858	4.594701	.0103093
98	9604	941,192	9.899495	4.610436	.0102041
99	9801	970,299	9.949874	4.626065	.0101010
100	10,000	1,000,000	10.000000	4.641589	.0100000

Logarithmic Cross-section Paper

Cross-section paper the rulings of which are proportional to the logarithms of the scale is called logarithmic cross-section paper. This paper is most convenient for plotting equations with constant exponents since they are straight lines on logarithmic paper while

they are curves if plotted on ordinary graph paper, in which case they must be plotted point by point.

The chief use of logarithmic cross-section paper is for plotting equations of the form:

$$y = ax^n$$

If two pairs of values of x and y are known, the corresponding points may be plotted on logarithmic paper and joined by a straight line. The value of the coefficient a is equal to the intercept of this line on the Y-axis, and the value of the exponent n is equal to the slope of the line (that is, the tangent of the angle which the line makes with the X-axis). The reason for this is that plotting on logarithmic paper is equivalent to taking logarithms, in which case we would obtain:

$$\log y = \log a + n \log x$$

which is the equation of a straight line, $\log a$ being the intercept and n the slope.

In case the values of a and n are known, that is, the intercept and the slope, we may plot the line, and from it obtain any pair of values of x and y.

Use of Logarithm Tables

Every logarithm consists of two parts: a positive or negative whole number called the **characteristic,** and a **positive** fraction, called the **mantissa.** The mantissa is always expressed as a decimal, and is the part which is given in the tables.

To find the common logarithm of a given number:

If the number is greater than 1, the characteristic of the logarithm is one unit less than the number of figures on the left of the decimal point.

If the number is less than 1, the characteristic of the logarithm is negative, and one unit more than the number of zeros between the decimal point and the first significant figure of the given number.

Thus,

$$\begin{aligned}
\log 20.6 &= 1.3139 \qquad \text{(base 10)}\\
\log 2.06 &= 0.3139\\
\log 0.206 &= 0.3139 - 1 = 9.3139 - 10\\
\log 0.0206 &= 0.3139 - 2 = 8.3139 - 10
\end{aligned}$$

To find the number corresponding to a given common logarithm:

If the characteristic of a given logarithm is positive, the number of figures in the integral part of the corresponding number is one more than the number of units in the characteristic.

If the characteristic is negative, the number of zeros between the decimal point and the first significant figure of the corresponding number is one less than the number of units in the characteristic.

COMMON LOGARITHMS OF NUMBERS
(Base 10)

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

COMMON LOGARITHMS OF NUMBERS

(Continued)

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

NATURAL LOGARITHMS OF NUMBERS FROM 1 TO 10 (Base *e*)

N	0	1	2	3	4	5	6	7	8	9
1.0	0.0000	0.0099	0.0198	0.0296	0.0392	0.0488	0.0583	0.0677	0.0770	0.0862
1.1	0.0953	0.1044	0.1133	0.1222	0.1310	0.1398	0.1484	0.1570	0.1655	0.1740
1.2	0.1823	0.1906	0.1989	0.2070	0.2151	0.2231	0.2311	0.2390	0.2469	0.2546
1.3	0.2624	0.2700	0.2776	0.2852	0.2927	0.3001	0.3075	0.3148	0.3221	0.3293
1.4	0.3365	0.3436	0.3507	0.3577	0.3646	0.3716	0.3784	0.3853	0.3920	0.3988
1.5	0.4055	0.4121	0.4187	0.4253	0.4318	0.4383	0.4447	0.4511	0.4574	0.4637
1.6	0.4700	0.4762	0.4824	0.4886	0.4947	0.5008	0.5068	0.5128	0.5188	0.5247
1.7	0.5306	0.5365	0.5423	0.5481	0.5539	0.5596	0.5653	0.5710	0.5766	0.5822
1.8	0.5878	0.5933	0.5988	0.6043	0.6098	0.6152	0.6206	0.6258	0.6313	0.6366
1.9	0.6419	0.6471	0.6523	0.6575	0.6627	0.6678	0.6729	0.6780	0.6831	0.6881
2.0	0.6932	0.6981	0.7031	0.7080	0.7130	0.7178	0.7227	0.7276	0.7324	0.7372
2.1	0.7419	0.7467	0.7514	0.7561	0.7608	0.7655	0.7701	0.7747	0.7793	0.7839
2.2	0.7885	0.7930	0.7975	0.8020	0.8065	0.8109	0.8154	0.8198	0.8242	0.8286
2.3	0.8329	0.8373	0.8416	0.8459	0.8502	0.8544	0.8587	0.8629	0.8671	0.8713
2.4	0.8755	0.8796	0.8838	0.8879	0.8920	0.8961	0.9001	0.9042	0.9083	0.9123
2.5	0.9163	0.9203	0.9243	0.9282	0.9322	0.9361	0.9400	0.9439	0.9478	0.9517
2.6	0.9555	0.9594	0.9632	0.9670	0.9708	0.9746	0.9783	0.9820	0.9858	0.9895
2.7	0.9933	0.9970	1.0006	1.0043	1.0080	1.0116	1.0152	1.0189	1.0225	1.0260
2.8	1.0296	1.0332	1.0367	1.0403	1.0438	1.0473	1.0508	1.0543	1.0578	1.0613
2.9	1.0647	1.0681	1.0716	1.0750	1.0784	1.0818	1.0852	1.0886	1.0919	1.0953
3.0	1.0986	1.1019	1.1053	1.1086	1.1119	1.1151	1.1184	1.1217	1.1249	1.1282
3.1	1.1314	1.1346	1.1378	1.1410	1.1442	1.1474	1.1506	1.1537	1.1569	1.1600
3.2	1.1632	1.1663	1.1694	1.1725	1.1756	1.1787	1.1817	1.1848	1.1878	1.1909
3.3	1.1939	1.1970	1.2000	1.2030	1.2060	1.2090	1.2119	1.2149	1.2179	1.2208
3.4	1.2238	1.2267	1.2296	1.2326	1.2355	1.2384	1.2413	1.2442	1.2470	1.2499
3.5	1.2528	1.2556	1.2585	1.2613	1.2641	1.2670	1.2698	1.2726	1.2754	1.2782
3.6	1.2809	1.2837	1.2865	1.2892	1.2920	1.2947	1.2975	1.3002	1.3029	1.3056
3.7	1.3083	1.3110	1.3137	1.3164	1.3191	1.3218	1.3244	1.3271	1.3297	1.3324
3.8	1.3350	1.3376	1.3403	1.3429	1.3455	1.3481	1.3507	1.3533	1.3558	1.3584
3.9	1.3610	1.3635	1.3661	1.3686	1.3712	1.3737	1.3762	1.3788	1.3813	1.3838
4.0	1.3863	1.3888	1.3913	1.3938	1.3962	1.3987	1.4012	1.4036	1.4061	1.4085
4.1	1.4110	1.4134	1.4159	1.4183	1.4207	1.4231	1.4255	1.4279	1.4303	1.4327
4.2	1.4351	1 4375	1.4398	1.4422	1.4446	1.4469	1.4493	1.4516	1.4540	1.4563
4.3	1.4586	1.4609	1.4633	1.4656	1.4679	1.4702	1.4725	1.4748	1.4770	1.4793
4.4	1.4816	1.4839	1.4861	1.4884	1.4907	1.4929	1.4951	1.4974	1.4996	1.5019
4.5	1.5041	1.5063	1.5085	1.5107	1.5129	1.5151	1.5173	1.5195	1.5217	1.5239
4.6	1.5261	1.5282	1.5304	1.5326	1.5347	1.5369	1.5390	1.5412	1.5433	1.5454
4.7	1.5476	1.5497	1.5518	1.5539	1.5560	1.5581	1.5603	1.5624	1.5644	1.5665
4.8	1.5686	1.5707	1.5728	1.5749	1.5769	1.5790	1.5810	1.5831	1.5852	1.5872
4.9	1.5892	1.5913	1.5933	1.5953	1.5974	1.5994	1.6014	1.6034	1.6054	1.6074
5.0	1.6094	1.6114	1.6134	1.6154	1.6174	1.6194	1.6214	1.6233	1.6253	1.6273
5.1	1.6292	1.6312	1.6332	1.6351	1.6371	1.6390	1.6409	1.6429	1.6448	1.6467
5.2	1.6487	1.6506	1.6525	1.6545	1.6563	1.6582	1.6601	1.6620	1.6639	1.6658
5.3	1.6677	1.6696	1.6715	1.6734	1.6753	1.6771	1.6790	1.6808	1.6827	1.6846
5.4	1.6864	1.6883	1.6901	1.6919	1.6938	1.6956	1.6975	1.6993	1.7011	1.7029

NATURAL LOGARITHMS OF NUMBERS

(Continued)

N	0	1	2	3	4	5	6	7	8	9
5.5	1.7048	1.7066	1.7084	1.7102	1.7120	1.7138	1.7156	1.7174	1.7192	1.7210
5.6	1.7228	1.7246	1.7263	1.7281	1.7299	1.7317	1.7334	1.7352	1.7370	1.7387
5.7	1.7405	1.7422	1.7440	1.7457	1.7475	1.7491	1.7509	1.7527	1.7544	1.7561
5.8	1.7579	1.7596	1.7613	1.7630	1.7647	1.7664	1.7682	1.7699	1.7716	1.7733
5.9	1.7750	1.7767	1.7783	1.7800	1.7817	1.7834	1.7851	1.7868	1.7884	1.7901
6.0	1.7918	1.7934	1.7951	1.7968	1.7984	1.8001	1.8017	1.8034	1.8050	1.8067
6.1	1.8083	1.8099	1.8116	1.8132	1.8148	1.8165	1.8181	1.8197	1.8213	1.8229
6.2	1.8246	1.8262	1.8278	1.8294	1.8310	1.8326	1.8342	1.8358	1.8374	1.8390
6.3	1.8406	1.8421	1.8437	1.8453	1.8469	1.8485	1.8500	1.8516	1.8532	1.8547
6.4	1.8563	1.8579	1.8594	1.8610	1.8625	1.8641	1.8656	1.8672	1.8687	1.8703
6.5	1.8718	1.8733	1.8749	1.8764	1.8779	1.8795	1.8810	1.8825	1.8840	1.8856
6.6	1.8871	1.8886	1.8901	1.8916	1.8931	1.8946	1.8961	1.8976	1.8991	1.9006
6.7	1.9021	1.9036	1.9051	1.9066	1.9081	1.9095	1.9110	1.9125	1.9140	1.9155
6.8	1.9169	1.9184	1.9199	1.9213	1.9228	1.9243	1.9257	1.9272	1.9286	1.9301
6.9	1.9315	1.9330	1.9344	1.9359	1.9373	1.9387	1.9402	1.9416	1.9431	1.9445
7.0	1.9459	1.9473	1.9488	1.9502	1.9516	1.9530	1.9545	1.9559	1.9573	1.9587
7.1	1.9601	1.9615	1.9629	1.9643	1.9657	1.9671	1.9685	1.9699	1.9713	1.9727
7.2	1.9741	1.9755	1.9769	1.9782	1.9796	1.9810	1.9824	1.9838	1.9851	1.9865
7.3	1.9879	1.9892	1.9906	1.9920	1.9933	1.9947	1.9961	1.9974	1.9988	2.0001
7.4	2.0015	2.0028	2.0042	2.0055	2.0069	2.0082	2.0096	2.0109	2.0122	2.0136
7.5	2.0149	2.0162	2.0176	2.0189	2.0202	2.0216	2.0229	2.0242	2.0255	2.0268
7.6	2.0282	2.0295	2.0308	2.0321	2.0334	2.0347	2.0360	2.0373	2.0386	2.0399
7.7	2.0412	2.0425	2.0438	2.0451	2.0464	2.0477	2.0490	2.0503	2.0516	2.0528
7.8	2.0541	2.0554	2.0567	2.0580	2.0592	2.0605	2.0618	2.0631	2.0643	2.0656
7.9	2.0669	2.0681	2.0694	2.0707	2.0719	2.0732	2.0744	2.0757	2.0769	2.0782
8.0	2.0794	2.0807	2.0819	2.0832	2.0844	2.0857	2.0869	2.0882	2.0894	2.0906
8.1	2.0919	2.0931	2.0943	2.0956	2.0968	2.0980	2.0992	2.1005	2.1017	2.1029
8.2	2.1041	2.1054	2.1066	2.1078	2.1090	2.1102	2.1114	2.1126	2.1138	2.1151
8.3	2.1163	2.1175	2.1187	2.1199	2.1211	2.1223	2.1235	2.1247	2.1259	2.1270
8.4	2.1282	2.1294	2.1306	2.1318	2.1330	2.1342	2.1354	2.1365	2.1377	2.1389
8.5	2.1401	2.1412	2.1424	2.1436	2.1448	2.1459	2.1471	2.1483	2.1494	2.1506
8.6	2.1518	2.1529	2.1541	2.1552	2.1564	2.1576	2.1587	2.1599	2.1610	2.1622
8.7	2.1633	2.1645	2.1656	2.1668	2.1679	2.1691	2.1702	2.1713	2.1725	2.1736
8.8	2.1748	2.1759	2.1770	2.1782	2.1793	2.1804	2.1816	2.1827	2.1838	2.1849
8.9	2.1861	2.1872	2.1883	2.1894	2.1905	2.1917	2.1928	2.1939	2.1950	2.1961
9.0	2.1972	2.1983	2.1994	2.2006	2.2017	2.2028	2.2039	2.2050	2.2061	2.2072
9.1	2.2083	2.2094	2.2105	2.2116	2.2127	2.2138	2.2149	2.2159	2.2170	2.2181
9.2	2.2192	2.2203	2.2214	2.2225	2.2235	2.2246	2.2257	2.2268	2.2279	2.2289
9.3	2.2300	2.2311	2.2322	2.2332	2.2343	2.2354	2.2365	2.2375	2.2386	2.2397
9.4	2.2407	2.2418	2.2428	2.2439	2.2450	2.2460	2.2471	2.2481	2.2492	2.2502
9.5	2.2513	2.2523	2.2534	2.2544	2.2555	2.2565	2.2576	2.2586	2.2597	2.2607
9.6	2.2618	2.2628	2.2638	2.2649	2.2659	2.2670	2.2680	2.2690	2.2701	2.2711
9.7	2.2721	2.2732	2.2742	2.2752	2.2762	2.2773	2.2783	2.2793	2.2803	2.2814
9.8	2.2824	2.2834	2.2844	2.2854	2.2865	2.2875	2.2885	2.2895	2.2905	2.2915
9.9	2.2925	2.2935	2.2946	2.2956	2.2966	2.2976	2.2986	2.2996	2.3006	2.3016

NATURAL LOGARITHMS (EACH INCREASED BY 10) OF NUMBERS FROM 0.00 TO 0.99

No.	0	1	2	3	4	5	6	7	8	9
0.0	....	5.395	6.088	6.493	6.781	7.004	7.187	7.341	7.474	7.592
0.1	7.697	7.793	7.880	7.960	8.034	8.103	8.167	8.228	8.285	8.339
0.2	8.391	8.439	8.486	8.530	8.573	8.614	8.653	8.691	8.727	8.762
0.3	8.796	8.829	8.861	8.891	8.921	8.950	8.978	9.006	9.032	9.058
0.4	9.084	9.108	9.132	9.156	9.179	9.201	9.223	9.245	9.266	9.287
0.5	9.307	9.327	9.346	9.365	9.384	9.402	9.420	9.438	9.455	9.472
0.6	9.489	9.506	9.522	9.538	9.554	9.569	9.584	9.600	9.614	9.629
0.7	9.643	9.658	9.671	9.685	9.699	9.712	9.726	9.739	9.752	9.764
0.8	9.777	9.789	9.802	9.814	9.826	9.837	9.849	9.861	9.872	9.883
0.9	9.895	9.906	9.917	9.927	9.938	9.949	9.959	9.970	9.980	9.990

NATURAL LOGARITHMS OF WHOLE NUMBERS FROM 10 TO 209

No.	0	1	2	3	4	5	6	7	8	9
1	2.303	2.398	2.485	2.565	2.639	2.708	2.773	2.833	2.890	2.944
2	2.996	3.045	3.091	3.136	3.178	3.219	3.258	3.296	3.332	3.367
3	3.401	3.434	3.466	3.497	3.526	3.555	3.584	3.611	3.638	3.664
4	3.689	3.714	3.738	3.761	3.784	3.807	3.829	3.850	3.871	3.892
5	3.912	3.932	3.951	3.970	3.989	4.007	4.025	4.043	4.060	4.078
6	4.094	4.111	4.127	4.143	4.159	4.174	4.190	4.205	4.220	4.234
7	4.249	4.263	4.277	4.291	4.304	4.318	4.331	4.344	4.357	4.369
8	4.382	4.394	4.407	4.419	4.431	4.443	4.454	4.466	4.477	4.489
9	4.500	4.511	4.522	4.533	4.543	4.554	4.564	4.575	4.585	4.595
10	4.605	4.615	4.625	4.635	4.644	4.654	4.663	4.673	4.682	4.691
11	4.701	4.710	4.719	4.727	4.736	4.745	4.754	4.762	4.771	4.779
12	4.788	4.796	4.804	4.812	4.820	4.828	4.836	4.844	4.852	4.860
13	4.868	4.875	4.883	4.890	4.898	4.905	4.913	4.920	4.927	4.935
14	4.942	4.949	4.956	4.963	4.970	4.977	4.984	4.990	4.997	5.004
15	5.011	5.017	5.024	5.030	5.037	5.043	5.050	5.056	5.063	5.069
16	5.075	5.081	5.088	5.094	5.100	5.106	5.112	5.118	5.124	5.130
17	5.136	5.142	5.148	5.153	5.159	5.165	5.171	5.176	5.182	5.187
18	5.193	5.199	5.204	5.210	5.215	5.220	5.226	5.231	5.236	5.242
19	5.247	5.252	5.258	5.263	5.268	5.273	5.278	5.283	5.288	5.293
20	5.298	5.303	5.308	5.313	5.318	5.323	5.328	5.333	5.338	5.342

LOGARITHMIC SINES, COSINES, TANGENTS, AND COTANGENTS

Degrees	sin	cos	tan	cot	
0° 00′	−∞	10.0000	−∞	+∞	90° 00′
0° 10′	7.4637	9.9999	7.4637	2.5363	89° 50′
0° 20′	7.7648	9.9999	7.7648	2.2352	89° 40′
0° 30′	7.9408	9.9999	7.9409	2.0591	89° 30′
0° 40′	8.0658	9.9999	8.0658	1.9342	89° 20′
0° 50′	8.1627	9.9999	8.1627	1.8373	89° 10′
1° 00′	8.2419	9.9999	8.2419	1.7581	89° 00′
1° 10′	8.3088	9.9999	8.3089	1.6911	88° 50′
1° 20′	8.3668	9.9999	8.3669	1.6331	88° 40′
1° 30′	8.4179	9.9999	8.4181	1.5819	88° 30′
1° 40′	8.4637	9.9998	8.4638	1.5362	88° 20′
1° 50′	8.5050	9.9998	8.5053	1.4947	88° 10′
2° 00′	8.5428	9.9997	8.5431	1.4569	88° 00′
2° 10′	8.5776	9.9997	8.5779	1.4221	87° 50′
2° 20′	8.6097	9.9996	8.6101	1.3899	87° 40′
2° 30′	8.6397	9.9996	8.6401	1.3599	87° 30′
2° 40′	8.6677	9.9995	8.6682	1.3318	87° 20′
2° 50′	8.6940	9.9995	8.6945	1.3055	87° 10′
3° 00′	8.7188	9.9994	8.7194	1.2806	87° 00′
3° 10′	8.7423	9.9993	8.7429	1.2571	86° 50′
3° 20′	8.7645	9.9993	8.7652	1.2348	86° 40′
3° 30′	8.7857	9.9992	8.7865	1.2135	86° 30′
3° 40′	8.8059	9.9991	8.8067	1.1933	86° 20′
3° 50′	8.8251	9.9990	8.8261	1.1739	86° 10′
4° 00′	8.8436	9.9989	8.8446	1.1554	86° 00′
4° 10′	8.8613	9.9989	8.8624	1.1376	85° 50′
4° 20′	8.8783	9.9988	8.8795	1.1205	85° 40′
4° 30′	8.8946	9.9987	8.8960	1.1040	85° 30′
4° 40′	8.9104	9.9986	8.9118	1.0882	85° 20′
4° 50′	8.9256	9.9985	8.9272	1.0728	85° 10′
5° 00′	8.9403	9.9983	8.9420	1.0580	85° 00′
5° 10′	8.9545	9.9982	8.9563	1.0437	84° 50′
5° 20′	8.9682	9.9981	8.9701	1.0299	84° 40′
5° 30′	8.9816	9.9980	8.9836	1.0164	84° 30′
	cos	sin	cot	tan	Degrees

LOGARITHMIC SINES, COSINES, TANGENTS, AND COTANGENTS (*Continued*)

Degrees	sin	cos	tan	cot	
5° 40′	8.9945	9.9979	8.9966	1.0034	84° 20′
5° 50′	9.0070	9.9977	9.0093	0.9907	84° 10′
6° 00′	9.0192	9.9976	9.0216	0.9784	84° 00′
6° 10′	9.0311	9.9975	9.0336	0.9664	83° 50′
6° 20′	9.0426	9.9973	9.0453	0.9547	83° 40′
6° 30′	9.0539	9.9972	9.0567	0.9433	83° 30′
6° 40′	9.0648	9.9971	9.0678	0.9322	83° 20′
6° 50′	9.0755	9.9969	9.0786	0.9214	83° 10′
7° 00′	9.0859	9.9968	9.0891	0.9109	83° 00′
7° 10′	9.0961	9.9966	9.0995	0.9005	82° 50′
7° 20′	9.1060	9.9964	9.1096	0.8904	82° 40′
7° 30′	9.1157	9.9963	9.1194	0.8806	82° 30′
7° 40′	9.1252	9.9961	9.1291	0.8709	82° 20′
7° 50′	9.1345	9.9959	9.1385	0.8615	82° 10′
8° 00′	9.1436	9.9958	9.1478	0.8522	82° 00′
8° 10′	9.1525	9.9956	9.1569	0.8431	81° 50′
8° 20′	9.1612	9.9954	9.1658	0.8342	81° 40′
8° 30′	9.1697	9.9952	9.1745	0.8255	81° 30′
8° 40′	9.1781	9.9950	9.1831	0.8169	81° 20′
8° 50′	9.1863	9.9948	9.1915	0.8085	81° 10′
9° 00′	9.1943	9.9946	9.1997	0.8003	81° 00′
9° 10′	9.2022	9.9944	9.2078	0.7922	80° 50′
9° 20′	9.2100	9.9942	9.2158	0.7842	80° 40′
9° 30′	9.2176	9.9940	9.2236	0.7764	80° 30′
9° 40′	9.2251	9.9938	9.2313	0.7687	80° 20′
9° 50′	9.2324	9.9936	9.2389	0.7611	80° 10′
10° 00′	9.2397	9.9934	9.2463	0.7537	80° 00′
10° 10′	9.2468	9.9931	9.2536	0.7464	79° 50′
10° 20′	9.2538	9.9929	9.2609	0.7391	79° 40′
10° 30′	9.2606	9.9927	9.2680	0.7320	79° 30′
10° 40′	9.2674	9.9924	9.2750	0.7250	79° 20′
10° 50′	9.2740	9.9922	9.2819	0.7181	79° 10′
11° 00′	9.2806	9.9919	9.2887	0.7113	79° 00′
11° 10′	9.2870	9.9917	9.2953	0.7047	78° 50′
	cos	sin	cot	tan	Degrees

LOGARITHMIC SINES, COSINES, TANGENTS, AND COTANGENTS (*Continued*)

Degrees	sin	cos	tan	cot	
11° 20′	9.2934	9.9914	9.3020	0.6980	78° 40′
11° 30′	9.2997	9.9912	9.3085	0.6915	78° 30′
11° 40′	9.3058	9.9909	9.3149	0.6851	78° 20′
11° 50′	9.3119	9.9907	9.3212	0.6788	78° 10′
12° 00′	9.3179	9.9904	9.3275	0.6725	78° 00′
12° 10′	9.3238	9.9901	9.3336	0.6664	77° 50′
12° 20′	9.3296	9.9899	9.3397	0.6603	77° 40′
12° 30′	9.3353	9.9896	9.3458	0.6542	77° 30′
12° 40′	9.3410	9.9893	9.3517	0.6483	77° 20′
12° 50′	9.3466	9.9890	9.3576	0.6424	77° 10′
13° 00′	9.3521	9.9887	9.3634	0.6366	77° 00′
13° 10′	9.3575	9.9884	9.3691	0.6309	76° 50′
13° 20′	9.3629	9.9881	9.3748	0.6252	76° 40′
13° 30′	9.3682	9.9878	9.3804	0.6196	76° 30′
13° 40′	9.3734	9.9875	9.3859	0.6141	76° 20′
13° 50′	9.3786	9.9872	9.3914	0.6086	76° 10′
14° 00′	9.3837	9.9869	9.3968	0.6032	76° 00′
14° 10′	9.3887	9.9866	9.4021	0.5979	75° 50′
14° 20′	9.3937	9.9863	9.4074	0.5926	75° 40′
14° 30′	9.3986	9.9859	9.4127	0.5873	75° 30′
14° 40′	9.4035	9.9856	9.4178	0.5822	75° 20′
14° 50′	9.4083	9.9853	9.4230	0.5770	75° 10′
15° 00′	9.4130	9.9849	9.4281	0.5719	75° 00′
15° 10′	9.4177	9.9846	9.4331	0.5669	74° 50′
15° 20′	9.4223	9.9843	9.4381	0.5619	74° 40′
15° 30′	9.4269	9.9839	9.4430	0.5570	74° 30′
15° 40′	9.4314	9.9836	9.4479	0.5521	74° 20′
15° 50′	9.4359	9.9832	9.4527	0.5473	74° 10′
16° 00′	9.4403	9.9828	9.4575	0.5425	74° 00′
16° 10′	9.4447	9.9825	9.4622	0.5378	73° 50′
16° 20′	9.4491	9.9821	9.4669	0.5331	73° 40′
16° 30′	9.4533	9.9817	9.4716	0.5284	73° 30′
16° 40′	9.4576	9.9814	9.4762	0.5238	73° 20′
16° 50′	9.4618	9.9810	9.4808	0.5192	73° 10′
	cos	sin	cot	tan	Degrees

LOGARITHMIC SINES, COSINES, TANGENTS, AND COTANGENTS (*Continued*)

Degrees	sin	cos	tan	cot	
17° 00′	9.4659	9.9806	9.4853	0.5147	73° 00′
17° 10′	9.4700	9.9802	9.4898	0.5102	72° 50′
17° 20′	9.4741	9.9798	9.4943	0.5057	72° 40′
17° 30′	9.4781	9.9794	9.4987	0.5013	72° 30′
17° 40′	9.4821	9.9790	9.5031	0.4969	72° 20′
17° 50′	9.4861	9.9786	9.5075	0.4925	72° 10′
18° 00′	9.4900	9.9782	9.5118	0.4882	72° 00′
18° 10′	9.4939	9.9778	9.5161	0.4839	71° 50′
18° 20′	9.4977	9.9774	9.5203	0.4797	71° 40′
18° 30′	9.5015	9.9770	9.5245	0.4755	71° 30′
18° 40′	9.5052	9.9765	9.5287	0.4713	71° 20′
18° 50′	9.5090	9.9761	9.5329	0.4671	71° 10′
19° 00′	9.5126	9.9757	9.5370	0.4630	71° 00′
19° 10′	9.5163	9.9752	9.5411	0.4589	70° 50′
19° 20′	9.5199	9.9748	9.5451	0.4549	70° 40′
19° 30′	9.5235	9.9743	9.5491	0.4509	70° 30′
19° 40′	9.5270	9.9739	9.5531	0.4469	70° 20′
19° 50′	9.5306	9.9734	9.5571	0.4429	70° 10′
20° 00′	9.5341	9.9730	9.5611	0.4389	70° 00′
20° 10′	9.5375	9.9725	9.5650	0.4350	69° 50′
20° 20′	9.5409	9.9721	9.5689	0.4311	69° 40′
20° 30′	9.5443	9.9716	9.5727	0.4273	69° 30′
20° 40′	9.5477	9.9711	9.5766	0.4234	69° 20′
20° 50′	9.5510	9.9706	9.5804	0.4196	69° 10′
21° 00′	9.5543	9.9702	9.5842	0.4158	69° 00′
21° 10′	9.5576	9.9697	9.5879	0.4121	68° 50′
21° 20′	9.5609	9.9692	9.5917	0.4083	68° 40′
21° 30′	9.5641	9.9687	9.5954	0.4046	68° 30′
21° 40′	9.5673	9.9682	9.5991	0.4009	68° 20′
21° 50′	9.5704	9.9677	9.6028	0.3972	68° 10′
22° 00′	9.5736	9.9672	9.6064	0.3936	68° 00′
22° 10′	9.5767	9.9667	9.6100	0.3900	67° 50′
22° 20′	9.5798	9.9661	9.6136	0.3864	67° 40′
22° 30′	9.5828	9.9656	9.6172	0.3828	67° 30′
	cos	sin	cot	tan	Degrees

LOGARITHMIC SINES, COSINES, TANGENTS, AND COTANGENTS (*Continued*)

Degrees	sin	cos	tan	cot	
22° 40′	9.5859	9.9651	9.6208	0.3792	67° 20′
22° 50′	9.5889	9.9646	9.6243	0.3757	67° 10′
23° 00′	9.5919	9.9640	9.6279	0.3721	67° 00′
23° 10′	9.5948	9.9635	9.6314	0.3686	66° 50′
23° 20′	9.5978	9.9629	9.6348	0.3652	66° 40′
23° 30′	9.6007	9.9624	9.6383	0.3617	66° 30′
23° 40′	9.6036	9.9618	9.6417	0.3583	66° 20′
23° 50′	9.6065	9.9613	9.6452	0.3548	66° 10′
24° 00′	9.6093	9.9607	9.6486	0.3514	66° 00′
24° 10′	9.6121	9.9602	9.6520	0.3480	65° 50′
24° 20′	9.6149	9.9596	9.6553	0.3447	65° 40′
24° 30′	9.6177	9.9590	9.6587	0.3413	65° 30′
24° 40′	9.6205	9.9584	9.6620	0.3380	65° 20′
24° 50′	9.6232	9.9579	9.6654	0.3346	65° 10′
25° 00′	9.6259	9.9573	9.6687	0.3313	65° 00′
25° 10′	9.6286	9.9567	9.6720	0.3280	64° 50′
25° 20′	9.6313	9.9561	9.6752	0.3248	64° 40′
25° 30′	9.6340	9.9555	9.6785	0.3215	64° 30′
25° 40′	9.6366	9.9549	9.6817	0.3183	64° 20′
25° 50′	9.6392	9.9543	9.6850	0.3150	64° 10′
26° 00′	9.6418	9.9537	9.6882	0.3118	64° 00′
26° 10′	9.6444	9.9530	9.6914	0.3086	63° 50′
26° 20′	9.6470	9.9524	9.6946	0.3054	63° 40′
26° 30′	9.6495	9.9518	9.6977	0.3023	63° 30′
26° 40′	9.6521	9.9512	9.7009	0.2991	63° 20′
26° 50′	9.6546	9.9505	9.7040	0.2960	63° 10′
27° 00′	9.6570	9.9499	9.7072	0.2928	63° 00′
27° 10′	9.6595	9.9492	9.7103	0.2897	62° 50′
27° 20′	9.6620	9.9486	9.7134	0.2866	62° 40′
27° 30′	9.6644	9.9479	9.7165	0.2835	62° 30′
27° 40′	9.6668	9.9473	9.7196	0.2804	62° 20′
27° 50′	9.6692	9.9466	9.7226	0.2774	62° 10′
28° 00′	9.6716	9.9459	9.7257	0.2743	62° 00′
28° 10′	9.6740	9.9453	9.7287	0.2713	61° 50′
	cos	sin	cot	tan	Degrees

LOGARITHMIC SINES, COSINES, TANGENTS, AND COTANGENTS (*Continued*)

Degrees	sin	cos	tan	cot	
28° 20′	9.6763	9.9446	9.7317	0.2683	61° 40′
28° 30′	9.6787	9.9439	9.7348	0.2652	61° 30′
28° 40′	9.6810	9.9432	9.7378	0.2622	61° 20′
28° 50′	9.6833	9.9425	9.7408	0.2592	61° 10′
29° 00′	9.6856	9.9418	9.7438	0.2562	61° 00′
29° 10′	9.6878	9.9411	9.7467	0.2533	60° 50′
29° 20′	9.6901	9.9404	9.7497	0.2503	60° 40′
29° 30′	9.6923	9.9397	9.7526	0.2474	60° 30′
29° 40′	9.6946	9.9390	9.7556	0.2444	60° 20′
29° 50′	9.6968	9.9383	9.7585	0.2415	60° 10′
30° 00′	9.6990	9.9375	9.7614	0.2386	60° 00′
30° 10′	9.7012	9.9368	9.7644	0.2356	59° 50′
30° 20′	9.7033	9.9361	9.7673	0.2327	59° 40′
30° 30′	9.7055	9.9353	9.7701	0.2299	59° 30′
30° 40′	9.7076	9.9346	9.7730	0.2270	59° 20′
30° 50′	9.7097	9.9338	9.7759	0.2241	59° 10′
31° 00′	9.7118	9.9331	9.7788	0.2212	59° 00′
31° 10′	9.7139	9.9323	9.7816	0.2184	58° 50′
31° 20′	9.7160	9.9315	9.7845	0.2155	58° 40′
31° 30′	9.7181	9.9308	9.7873	0.2127	58° 30′
31° 40′	9.7201	9.9300	9.7902	0.2098	58° 20′
31° 50′	9.7222	9.9292	9.7930	0.2070	58° 10′
32° 00′	9.7242	9.9284	9.7958	0.2042	58° 00′
32° 10′	9.7262	9.9276	9.7986	0.2014	57° 50′
32° 20′	9.7282	9.9268	9.8014	0.1986	57° 40′
32° 30′	9.7302	9.9260	9.8042	0.1958	57° 30′
32° 40′	9.7322	9.9252	9.8070	0.1930	57° 20′
32° 50′	9.7342	9.9244	9.8097	0.1903	57° 10′
33° 00′	9.7361	9.9236	9.8125	0.1875	57° 00′
33° 10′	9.7380	9.9228	9.8153	0.1847	56° 50′
33° 20′	9.7400	9.9219	9.8180	0.1820	56° 40′
33° 30′	9.7419	9.9211	9.8208	0.1792	56° 30′
33° 40′	9.7438	9.9203	9.8235	0.1765	56° 20′
33° 50′	9.7457	9.9194	9.8263	0.1737	56° 10′
	cos	sin	cot	tan	Degrees

LOGARITHMIC SINES, COSINES, TANGENTS, AND COTANGENTS (*Continued*)

Degrees	sin	cos	tan	cot	
34° 00′	9.7476	9.9186	9.8290	0.1710	56° 00′
34° 10′	9.7494	9.9177	9.8317	0.1683	55° 50′
34° 20′	9.7513	9.9169	9.8344	0.1656	55° 40′
34° 30′	9.7531	9.9160	9.8371	0.1629	55° 30′
34° 40′	9.7550	9.9151	9.8398	0.1602	55° 20′
34° 50′	9.7568	9.9142	9.8425	0.1575	55° 10′
35° 00′	9.7586	9.9134	9.8452	0.1548	55° 00′
35° 10′	9.7604	9.9125	9.8479	0.1521	54° 50′
35° 20′	9.7622	9.9116	9.8506	0.1494	54° 40′
35° 30′	9.7640	9.9107	9.8533	0.1467	54° 30′
35° 40′	9.7657	9.9098	9.8559	0.1441	54° 20′
35° 50′	9.7675	9.9089	9.8586	0.1414	54° 10′
36° 00′	9.7692	9.9080	9.8613	0.1387	54° 00′
36° 10′	9.7710	9.9070	9.8639	0.1361	53° 50′
36° 20′	9.7727	9.9061	9.8666	0.1334	53° 40′
36° 30′	9.7744	9.9052	9.8692	0.1308	53° 30′
36° 40′	9.7761	9.9042	9.8718	0.1282	53° 20′
36° 50′	9.7778	9.9033	9.8745	0.1255	53° 10′
37° 00′	9.7795	9.9023	9.8771	0.1229	53° 00′
37° 10′	9.7811	9.9014	9.8797	0.1203	52° 50′
37° 20′	9.7828	9.9004	9.8824	0.1176	52° 40′
37° 30′	9.7844	9.8995	9.8850	0.1150	52° 30′
37° 40′	9.7861	9.8985	9.8876	0.1124	52° 20′
37° 50′	9.7877	9.8975	9.8902	0.1098	52° 10′
38° 00′	9.7893	9.8965	9.8928	0.1072	52° 00′
38° 10′	9.7910	9.8955	9.8954	0.1046	51° 50′
38° 20′	9.7926	9.8945	9.8980	0.1020	51° 40′
38° 30′	9.7941	9.8935	9.9006	0.0994	51° 30′
38° 40′	9.7957	9.8925	9.9032	0.0968	51° 20′
38° 50′	9.7973	9.8915	9.9058	0.0942	51° 10′
39° 00′	9.7989	9.8905	9.9084	0.0916	51° 00′
39° 10′	9.8004	9.8895	9.9110	0.0890	50° 50′
39° 20′	9.8020	9.8884	9.9135	0.0865	50° 40′
39° 30′	9.8035	9.8874	9.9161	0.0839	50° 30′
39° 40′	9.8050	9.8864	9.9187	0.0813	50° 20′
39° 50′	9.8066	9.8853	9.9212	0.0788	50° 10′
	cos	sin	cot	tan	Degrees

LOGARITHMIC SINES, COSINES, TANGENTS, AND COTANGENTS (*Continued*)

Degrees	sin	cos	tan	cot	
40° 00′	9.8081	9.8843	9.9238	0.0762	50° 00′
40° 10′	9.8096	9.8832	9.9264	0.0736	49° 50′
40° 20′	9.8111	9.8821	9.9289	0.0711	49° 40′
40° 30′	9.8125	9.8810	9.9315	0.0685	49° 30′
40° 40′	9.8140	9.8800	9.9341	0.0659	49° 20′
40° 50′	9.8155	9.8789	9.9366	0.0634	49° 10′
41° 00′	9.8169	9.8778	9.9392	0.0608	49° 00′
41° 10′	9.8184	9.8767	9.9417	0.0583	48° 50′
41° 20′	9.8198	9.8756	9.9443	0.0557	48° 40′
41° 30′	9.8213	9.8745	9.9468	0.0532	48° 30′
41° 40′	9.8227	9.8733	9.9494	0.0506	48° 20′
41° 50′	9.8241	9.8722	9.9519	0.0481	48° 10′
42° 00′	9.8255	9.8711	9.9544	0.0456	48° 00′
42° 10′	9.8269	9.8699	9.9570	0.0430	47° 50′
42° 20′	9.8283	9.8688	9.9595	0.0405	47° 40′
42° 30′	9.8297	9.8676	9.9621	0.0379	47° 30′
42° 40′	9.8311	9.8665	9.9646	0.0354	47° 20′
42° 50′	9.8324	9.8653	9.9671	0.0329	47° 10′
43° 00′	9.8338	9.8641	9.9697	0.0303	47° 00′
43° 10′	9.8351	9.8629	9.9722	0.0278	46° 50′
43° 20′	9.8365	9.8618	9.9747	0.0253	46° 40′
43° 30′	9.8378	9.8606	9.9772	0.0228	46° 30′
43° 40′	9.8391	9.8594	9.9798	0.0202	46° 20′
43° 50′	9.8405	9.8582	9.9823	0.0177	46° 10′
44° 00′	9.8418	9.8569	9.9848	0.0152	46° 00′
44° 10′	9.8431	9.8557	9.9874	0.0126	45° 50′
44° 20′	9.8444	9.8545	9.9899	0.0101	45° 40′
44° 30′	9.8457	9.8532	9.9924	0.0076	45° 30′
44° 40′	9.8469	9.8520	9.9949	0.0051	45° 20′
44° 50′	9.8482	9.8507	9.9975	0.0025	45° 10′
45° 00′	9.8495	9.8495	0.0000	0.0000	45° 00′
	cos	sin	cot	tan	Degrees

NATURAL SINES, COSINES, TANGENTS, AND COTANGENTS

Degrees	sin	cos	tan	cot	
0° 00′	.0000	1.0000	.0000	∞	90° 00′
0° 10′	.0029	1.0000	.0029	343.77	89° 50′
0° 20′	.0058	1.0000	.0058	171.89	89° 40′
0° 30′	.0087	1.0000	.0087	114.59	89° 30′
0° 40′	.0116	.9999	.0116	85.940	89° 20′
0° 50′	.0145	.9999	.0145	68.750	89° 10′
1° 00′	.0175	.9998	.0175	57.290	89° 00′
1° 10′	.0204	.9998	.0204	49.104	88° 50′
1° 20′	.0233	.9997	.0233	42.964	88° 40′
1° 30′	.0262	.9997	.0262	38.188	88° 30′
1° 40′	.0291	.9996	.0291	34.368	88° 20′
1° 50′	.0320	.9995	.0320	31.242	88° 10′
2° 00′	.0349	.9994	.0349	28.636	88° 00′
2° 10′	.0378	.9993	.0378	26.432	87° 50′
2° 20′	.0407	.9992	.0407	24.542	87° 40′
2° 30′	.0436	.9990	.0437	22.904	87° 30′
2° 40′	.0465	.9989	.0466	21.470	87° 20′
2° 50′	.0494	.9988	.0495	20.206	87° 10′
3° 00′	.0523	.9986	.0524	19.081	87° 00′
3° 10′	.0552	.9985	.0553	18.075	86° 50′
3° 20′	.0581	.9983	.0582	17.169	86° 40′
3° 30′	.0610	.9981	.0612	16.350	86° 30′
3° 40′	.0640	.9980	.0641	15.605	86° 20′
3° 50′	.0669	.9978	.0670	14.924	86° 10′
4° 00′	.0698	.9976	.0699	14.301	86° 00′
4° 10′	.0727	.9974	.0729	13.727	85° 50′
4° 20′	.0756	.9971	.0758	13.197	85° 40′
4° 30′	.0785	.9969	.0787	12.706	85° 30′
4° 40′	.0814	.9967	.0816	12.251	85° 20′
4° 50′	.0843	.9964	.0846	11.826	85° 10′
5° 00′	.0872	.9962	.0875	11.430	85° 00′
5° 10′	.0901	.9959	.0904	11.059	84° 50′
5° 20′	.0929	.9957	.0934	10.712	84° 40′
5° 30′	.0958	.9954	.0963	10.385	84° 30′
	cos	sin	cot	tan	Degrees

NATURAL SINES, COSINES, TANGENTS, AND COTANGENTS (*Continued*)

Degrees	sin	cos	tan	cot	
5° 40′	.0987	.9951	.0992	10.078	84° 20′
5° 50′	.1016	.9948	.1022	9.7882	84° 10′
6° 00′	.1045	.9945	.1051	9.5144	84° 00′
6° 10′	.1074	.9942	.1080	9.2553	83° 50′
6° 20′	.1103	.9939	.1110	9.0098	83° 40′
6° 30′	.1132	.9936	.1139	8.7769	83° 30′
6° 40′	.1161	.9932	.1169	8.5555	83° 20′
6° 50′	.1190	.9929	.1198	8.3450	83° 10′
7° 00′	.1219	.9925	.1228	8.1443	83° 00′
7° 10′	.1248	.9922	.1257	7.9530	82° 50′
7° 20′	.1276	.9918	.1287	7.7704	82° 40′
7° 30′	.1305	.9914	.1317	7.5958	82° 30′
7° 40′	.1334	.9911	.1346	7.4287	82° 20′
7° 50′	.1363	.9907	.1376	7.2687	82° 10′
8° 00′	.1392	.9903	.1405	7.1154	82° 00′
8° 10′	.1421	.9899	.1435	6.9682	81° 50′
8° 20′	.1449	.9894	.1465	6.8269	81° 40′
8° 30′	.1478	.9890	.1495	6.6912	81° 30′
8° 40′	.1507	.9886	.1524	6.5606	81° 20′
8° 50′	.1536	.9881	.1554	6.4348	81° 10′
9° 00′	.1564	.9877	.1584	6.3138	81° 00′
9° 10′	.1593	.9872	.1614	6.1970	80° 50′
9° 20′	.1622	.9868	.1644	6.0844	80° 40′
9° 30′	.1650	.9863	.1673	5.9758	80° 30′
9° 40′	.1679	.9858	.1703	5.8708	80° 20′
9° 50′	.1708	.9853	.1733	5.7694	80° 10′
10° 00′	.1736	.9848	.1763	5.6713	80° 00′
10° 10′	.1765	.9843	.1793	5.5764	79° 50′
10° 20′	.1794	.9838	.1823	5.4845	79° 40′
10° 30′	.1822	.9833	.1853	5.3955	79° 30′
10° 40′	.1851	.9827	.1883	5.3093	79° 20′
10° 50′	.1880	.9822	.1914	5.2257	79° 10′
11° 00′	.1908	.9816	.1944	5.1446	79° 00′
11° 10′	.1937	.9811	.1974	5.0658	78° 50′
	cos	sin	cot	tan	Degrees

NATURAL SINES, COSINES, TANGENTS, AND COTANGENTS (*Continued*)

Degrees	sin	cos	tan	cot	
11° 20′	.1965	.9805	.2004	4.9894	78° 40′
11° 30′	.1994	.9799	.2035	4.9152	78° 30′
11° 40′	.2022	.9793	.2065	4.8430	78° 20′
11° 50′	.2051	.9787	.2095	4.7729	78° 10′
12° 00′	.2079	.9781	.2126	4.7046	78° 00′
12° 10′	.2108	.9775	.2156	4.6382	77° 50′
12° 20′	.2136	.9769	.2186	4.5736	77° 40′
12° 30′	.2164	.9763	.2217	4.5107	77° 30′
12° 40′	.2193	.9757	.2247	4.4494	77° 20′
12° 50′	.2221	.9750	.2278	4.3897	77° 10′
13° 00′	.2250	.9744	.2309	4.3315	77° 00′
13° 10′	.2278	.9737	.2339	4.2747	76° 50′
13° 20′	.2306	.9730	.2370	4.2193	76° 40′
13° 30′	.2334	.9724	.2401	4.1653	76° 30′
13° 40′	.2363	.9717	.2432	4.1126	76° 20′
13° 50′	.2391	.9710	.2462	4.0611	76° 10′
14° 00′	.2419	.9703	.2493	4.0108	76° 00′
14° 10′	.2447	.9696	.2524	3.9617	75° 50′
14° 20′	.2476	.9689	.2555	3.9136	75° 40′
14° 30′	.2504	.9681	.2586	3.8667	75° 30′
14° 40′	.2532	.9674	.2617	3.8208	75° 20′
14° 50′	.2560	.9667	.2648	3.7760	75° 10′
15° 00′	.2588	.9659	.2679	3.7321	75° 00′
15° 10′	.2616	.9652	.2711	3.6891	74° 50′
15° 20′	.2644	.9644	.2742	3.6470	74° 40′
15° 30′	.2672	.9636	.2773	3.6059	74° 30′
15° 40′	.2700	.9628	.2805	3.5656	74° 20′
15° 50′	.2728	.9621	.2836	3.5261	74° 10′
16° 00′	.2756	.9613	.2867	3.4874	74° 00′
16° 10′	.2784	.9605	.2899	3.4495	73° 50′
16° 20′	.2812	.9596	.2931	3.4124	73° 40′
16° 30′	.2840	.9588	.2962	3.3759	73° 30′
16° 40′	.2868	.9580	.2994	3.3402	73° 20′
16° 50′	.2896	.9572	.3026	3.3052	73° 10′
	cos	sin	cot	tan	Degrees

NATURAL SINES, COSINES, TANGENTS, AND COTANGENTS (*Continued*)

Degrees	sin	cos	tan	cot	
17° 00′	.2924	.9563	.3057	3.2709	73° 00′
17° 10′	.2952	.9555	.3089	3.2371	72° 50′
17° 20′	.2979	.9546	.3121	3.2041	72° 40′
17° 30′	.3007	.9537	.3153	3.1716	72° 30′
17° 40′	.3035	.9528	.3185	3.1397	72° 20′
17° 50′	.3062	.9520	.3217	3.1084	72° 10′
18° 00′	.3090	.9511	.3249	3.0777	72° 00′
18° 10′	.3118	.9502	.3281	3.0475	71° 50′
18° 20′	.3145	.9492	.3314	3.0178	71° 40′
18° 30′	.3173	.9483	.3346	2.9887	71° 30′
18° 40′	.3201	.9474	.3378	2.9600	71° 20′
18° 50′	.3228	.9465	.3411	2.9319	71° 10′
19° 00′	.3256	.9455	.3443	2.9042	71° 00′
19° 10′	.3283	.9446	.3476	2.8770	70° 50′
19° 20′	.3311	.9436	.3508	2.8502	70° 40′
19° 30′	.3338	.9426	.3541	2.8239	70° 30′
19° 40′	.3365	.9417	.3574	2.7980	70° 20′
19° 50′	.3393	.9407	.3607	2.7725	70° 10′
20° 00′	.3420	.9397	.3640	2.7475	70° 00′
20° 10′	.3448	.9387	.3673	2.7228	69° 50′
20° 20′	.3475	.9377	.3706	2.6985	69° 40′
20° 30′	.3502	.9367	.3739	2.6746	69° 30′
20° 40′	.3529	.9356	.3772	2.6511	69° 20′
20° 50′	.3557	.9346	.3805	2.6279	69° 10′
21° 00′	.3584	.9336	.3839	2.6051	69° 00′
21° 10′	.3611	.9325	.3872	2.5826	68° 50′
21° 20′	.3638	.9315	.3906	2.5605	68° 40′
21° 30′	.3665	.9304	.3939	2.5386	68° 30′
21° 40′	.3692	.9293	.3973	2.5172	68° 20′
21° 50′	.3719	.9283	.4006	2.4960	68° 10′
22° 00′	.3746	.9272	.4040	2.4751	68° 00′
22° 10′	.3773	.9261	.4074	2.4545	67° 50′
22° 20′	.3800	.9250	.4108	2.4342	67° 40′
22° 30′	.3827	.9239	.4142	2.4142	67° 30′
	cos	sin	cot	tan	Degrees

NATURAL SINES, COSINES, TANGENTS, AND COTANGENTS (*Continued*)

Degrees	sin	cos	tan	cot	
22° 40′	.3854	.9228	.4176	2.3945	67° 20′
22° 50′	.3881	.9216	.4210	2.3750	67° 10′
23° 00′	.3907	.9205	.4245	2.3559	67° 00′
23° 10′	.3934	.9194	.4279	2.3369	66° 50′
23° 20′	.3961	.9182	.4314	2.3183	66° 40′
23° 30′	.3987	.9171	.4348	2.2998	66° 30′
23° 40′	.4014	.9159	.4383	2.2817	66° 20′
23° 50′	.4041	.9147	.4417	2.2637	66° 10′
24° 00′	.4067	.9135	.4452	2.2460	66° 00′
24° 10′	.4094	.9124	.4487	2.2286	65° 50′
24° 20′	.4120	.9112	.4522	2.2113	65° 40′
24° 30′	.4147	.9100	.4557	2.1943	65° 30′
24° 40′	.4173	.9088	.4592	2.1775	65° 20′
24° 50′	.4200	.9075	.4628	2.1609	65° 10′
25° 00′	.4226	.9063	.4663	2.1445	65° 00′
25° 10′	.4253	.9051	.4699	2.1283	64° 50′
25° 20′	.4279	.9038	.4734	2.1123	64° 40′
25° 30′	.4305	.9026	.4770	2.0965	64° 30′
25° 40′	.4331	.9013	.4806	2.0809	64° 20′
25° 50′	.4358	.9001	.4841	2.0655	64° 10′
26° 00′	.4384	.8988	.4877	2.0503	64° 00′
26° 10′	.4410	.8975	.4913	2.0353	63° 50′
26° 20′	.4436	.8962	.4950	2.0204	63° 40′
26° 30′	.4462	.8949	.4986	2.0057	63° 30′
26° 40′	.4488	.8936	.5022	1.9912	63° 20′
26° 50′	.4514	.8923	.5059	1.9768	63° 10′
27° 00′	.4540	.8910	.5095	1.9626	63° 00′
27° 10′	.4566	.8897	.5132	1.9486	62° 50′
27° 20′	.4592	.8884	.5169	1.9347	62° 40′
27° 30′	.4617	.8870	.5206	1.9210	62° 30′
27° 40′	.4643	.8857	.5243	1.9074	62° 20′
27° 50′	.4669	.8843	.5280	1.8940	62° 10′
28° 00′	.4695	.8829	.5317	1.8807	62° 00′
28° 10′	.4720	.8816	.5354	1.8676	61° 50′
	cos	sin	cot	tan	Degrees

NATURAL SINES, COSINES, TANGENTS, AND COTANGENTS (*Continued*)

Degrees	sin	cos	tan	cot	
28° 20′	.4746	.8802	.5392	1.8546	61° 40′
28° 30′	.4772	.8788	.5430	1.8418	61° 30′
28° 40′	.4797	.8774	.5467	1.8291	61° 20′
28° 50′	.4823	.8760	.5505	1.8165	61° 10′
29° 00′	.4848	.8746	.5543	1.8040	61° 00′
29° 10′	.4874	.8732	.5581	1.7917	60° 50′
29° 20′	.4899	.8718	.5619	1.7796	60° 40′
29° 30′	.4924	.8704	.5658	1.7675	60° 30′
29° 40′	.4950	.8689	.5696	1.7556	60° 20′
29° 50′	.4975	.8675	.5735	1.7437	60° 10′
30° 00′	.5000	.8660	.5774	1.7321	60° 00′
30° 10′	.5025	.8646	.5812	1.7205	59° 50′
30° 20′	.5050	.8631	.5851	1.7090	59° 40′
30° 30′	.5075	.8616	.5890	1.6977	59° 30′
30° 40′	.5100	.8601	.5930	1.6864	59° 20′
30° 50′	.5125	.8587	.5969	1.6753	59° 10′
31° 00′	.5150	.8572	.6009	1.6643	59° 00′
31° 10′	.5175	.8557	.6048	1.6534	58° 50′
31° 20′	.5200	.8542	.6088	1.6426	58° 40′
31° 30′	.5225	.8526	.6128	1.6319	58° 30′
31° 40′	.5250	8511	.6168	1.6212	58° 20′
31° 50′	.5275	.8496	.6208	1.6107	58° 10′
32° 00′	.5299	.8480	.6249	1.6003	58° 00′
32° 10′	.5324	.8465	.6289	1.5900	57° 50′
32° 20′	.5348	.8450	.6330	1.5798	57° 40′
32° 30′	.5373	.8434	.6371	1.5697	57° 30′
32° 40′	.5398	.8418	.6412	1.5597	57° 20′
32° 50′	.5422	.8403	.6453	1.5497	57° 10′
33° 00′	.5446	.8387	.6494	1.5399	57° 00′
33° 10′	.5471	.8371	.6536	1.5301	56° 50′
33° 20′	.5495	.8355	.6577	1.5204	56° 40′
33° 30′	.5519	.8339	.6619	1.5108	56° 30′
33° 40′	.5544	.8323	.6661	1.5013	56° 20′
33° 50′	.5568	.8307	.6703	1.4919	56° 10′
	cos	sin	cot	tan	Degrees

NATURAL SINES, COSINES, TANGENTS, AND COTANGENTS (*Continued*)

Degrees	sin	cos	tan	cot	
34° 00′	.5592	.8290	.6745	1.4826	56° 00′
34° 10′	.5616	.8274	.6787	1.4733	55° 50′
34° 20′	.5640	.8258	.6830	1.4641	55° 40′
34° 30′	.5664	.8241	.6873	1.4550	55° 30′
34° 40′	.5688	.8225	.6916	1.4460	55° 20′
34° 50′	.5712	.8208	.6959	1.4370	55° 10′
35° 00′	.5736	.8192	.7002	1.4281	55° 00′
35° 10′	.5760	.8175	.7046	1.4193	54° 50′
35° 20′	.5783	.8158	.7089	1.4106	54° 40′
35° 30′	.5807	.8141	.7133	1.4019	54° 30′
35° 40′	.5831	.8124	.7177	1.3934	54° 20′
35° 50′	.5854	.8107	.7221	1.3848	54° 10′
36° 00′	.5878	.8090	.7265	1.3764	54° 00′
36° 10′	.5901	.8073	.7310	1.3680	53° 50′
36° 20′	.5925	.8056	.7355	1.3597	53° 40′
36° 30′	.5948	.8039	.7400	1.3514	53° 30′
36° 40′	.5972	.8021	.7445	1.3432	53° 20′
36° 50′	.5995	.8004	.7490	1.3351	53° 10′
37° 00′	.6018	.7986	.7536	1.3270	53° 00′
37° 10′	.6041	.7969	.7581	1.3190	52° 50′
37° 20′	.6065	.7951	.7627	1.3111	52° 40′
37° 30′	.6088	.7934	.7673	1.3032	52° 30′
37° 40′	.6111	.7916	.7720	1.2954	52° 20′
37° 50′	.6134	.7898	.7766	1.2876	52° 10′
38° 00′	.6157	.7880	.7813	1.2799	52° 00′
38° 10′	.6180	.7862	.7860	1.2723	51° 50′
38° 20′	.6202	.7844	.7907	1.2647	51° 40′
38° 30′	.6225	.7826	.7954	1.2572	51° 30′
38° 40′	.6248	.7808	.8002	1.2497	51° 20′
38° 50′	.6271	.7790	.8050	1.2423	51° 10′
39° 00′	.6293	.7771	.8098	1.2349	51° 00′
39° 10′	.6316	.7753	.8146	1.2276	50° 50′
39° 20′	.6338	.7735	.8195	1.2203	50° 40′
39° 30′	.6361	.7716	.8243	1.2131	50° 30′
	cos	sin	cot	tan	Degrees

NATURAL SINES, COSINES, TANGENTS, AND COTANGENTS (*Continued*)

Degrees	sin	cos	tan	cot	
39° 40′	.6383	.7698	.8292	1.2059	50° 20′
39° 50′	.6406	.7679	.8342	1.1988	50° 10′
40° 00′	.6428	.7660	.8391	1.1918	50° 00′
40° 10′	.6450	.7642	.8441	1.1847	49° 50′
40° 20′	.6472	.7623	.8491	1.1778	49° 40′
40° 30′	.6494	.7604	.8541	1.1708	49° 30′
40° 40′	.6517	.7585	.8591	1.1640	49° 20′
40° 50′	.6539	.7566	.8642	1.1571	49° 10′
41° 00′	.6561	.7547	.8693	1.1504	49° 00′
41° 10′	.6583	.7528	.8744	1.1436	48° 50′
41° 20′	.6604	.7509	.8796	1.1369	48° 40′
41° 30′	.6626	.7490	.8847	1.1303	48° 30′
41° 40′	.6648	.7470	.8899	1.1237	48° 20′
41° 50′	.6670	.7451	.8952	1.1171	48° 10′
42° 00′	.6691	.7431	.9004	1.1106	48° 00′
42° 10′	.6713	.7412	.9057	1.1041	47° 50′
42° 20′	.6734	.7392	.9110	1.0977	47° 40′
42° 30′	.6756	.7373	.9163	1.0913	47° 30′
42° 40′	.6777	.7353	.9217	1.0850	47° 20′
42° 50′	.6799	.7333	.9271	1.0786	47° 10′
43° 00′	.6820	.7314	.9325	1.0724	47° 00′
43° 10′	.6841	.7294	.9380	1.0661	46° 50′
43° 20′	.6862	.7274	.9435	1.0599	46° 40′
43° 30′	.6884	.7254	.9490	1.0538	46° 30′
43° 40′	.6905	.7234	.9545	1.0477	46° 20′
43° 50′	.6926	.7214	.9601	1.0416	46° 10′
44° 00′	.6947	.7193	.9657	1.0355	46° 00′
44° 10′	.6967	.7173	.9713	1.0295	45° 50′
44° 20′	.6988	.7153	.9770	1.0235	45° 40′
44° 30′	.7009	.7133	.9827	1.0176	45° 30′
44° 40′	.7030	.7112	.9884	1.0117	45° 20′
44° 50′	.7050	.7092	.9942	1.0058	45° 10′
45° 00′	.7071	.7071	1.0000	1.0000	45° 00′
	cos	sin	cot	tan	Degrees

HYPERBOLIC SINES AND COSINES

n	cosh n	sinh n	n	cosh n	sinh n
0.00	1.0000	0.0000	2.05	3.9484	3.8196
0.05	1.0013	0.0500	2.10	4.1443	4.0219
0.10	1.0050	0.1002	2.15	4.3507	4.2342
0.15	1.0112	0.1506	2.20	4.5679	4.4571
0.20	1.0201	0.2013	2.25	4.7966	4.6912
0.25	1.0314	0.2526	2.30	5.0372	4.9369
0.30	1.0453	0.3045	2.35	5.2905	5.1952
0.35	1.0619	0.3572	2.40	5.5569	5.4662
0.40	1.0811	0.4108	2.45	5.8373	5.7510
0.45	1.1030	0.4653	2.50	6.1323	6.0502
0.50	1.1276	0.5211	2.55	6.4426	6.3645
0.55	1.1551	0.5782	2.60	6.7690	6.6947
0.60	1.1855	0.6367	2.65	7.1123	7.0417
0.65	1.2188	0.6967	2.70	7.4735	7.4063
0.70	1.2552	0.7586	2.75	7.8533	7.7894
0.75	1.2947	0.8223	2.80	8.2527	8.1919
0.80	1.3374	0.8881	2.85	8.6728	8.6150
0.85	1.3835	0.9561	2.90	9.1146	9.0596
0.90	1.4331	1.0265	2.95	9.5791	9.5268
0.95	1.4862	1.0995	3.00	10.0677	10.0179
1.00	1.5431	1.1752	3.05	10.5814	10.5340
1.05	1.6038	1.2539	3.10	11.1215	11.0765
1.10	1.6685	1.3356	3.15	11.6895	11.6466
1.15	1.7374	1.4208	3.20	12.2866	12.2459
1.20	1.8107	1.5097	3.25	12.9146	12.8758
1.25	1.8884	1.6019	3.30	13.5748	13.5379
1.30	1.9709	1.6984	3.35	14.2689	14.2338
1.35	2.0583	1.7991	3.40	14.9987	14.9654
1.40	2.1509	1.9043	3.45	15.7661	15.7343
1.45	2.2488	2.0143	3.50	16.5728	16.5426
1.50	2.3524	2.1293	3.55	17.4210	17.3923
1.55	2.4619	2.2496	3.60	18.3128	18.2855
1.60	2.5775	2.3757	3.65	19.2503	19.2243
1.65	2.6995	2.5075	3.70	20.2360	20.2113
1.70	2.8283	2.6456	3.75	21.2723	21.2488
1.75	2.9642	2.7904	3.80	22.3618	22.3394
1.80	3.1075	2.9422	3.85	23.5072	23.4859
1.85	3.2583	3.1013	3.90	24.7113	24.6911
1.90	3.4177	3.2682	3.95	25.9773	25.9581
1.95	3.5855	3.4432	4.00	27.3082	27.2899
2.00	3.7622	3.6269	....	......	......

Numerical Constants

$$\pi = 3.141\ 592\ 654$$

$$\log_{10} \pi = 0.497\ 149\ 873$$

$$\frac{1}{\pi} = 0.318\ 309\ 886$$

$$\pi^2 = 9.869\ 604\ 401$$

$$\sqrt{\pi} = 1.772\ 453\ 851$$

$$e = 2.718\ 281\ 828$$

$$\log_{10} e = 0.434\ 294\ 482$$

$$\log_e 10 = 2.302\ 585\ 093$$

$$\log_{10} \log_{10} e = 9.637\ 784\ 311$$

$$\log_e \pi = 1.144\ 729\ 886$$

$$\log_e 2 = 0.693\ 147\ 181$$

$$\log_{10} 2 = 0.301\ 029\ 996$$

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